Thailand Team Selection Test for IMO 2020
IPST, Bangkok
21 January 2020

TST 1 Day 1
Time: 4.5 hours

1. Let ABC be an acute-angled triangle and let D, E, and F be the feet of altitudes from A, B, and C to sides BC, CA, and AB, respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE, and let these circles be tangent to segments DF and DE at M and N, respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively. Prove that $MP = NQ$.

2. Alice has a map of Wonderland, a country consisting of $n \geq 2$ towns. For every pair of towns, there is a narrow road going from one town to the other. One day, all the roads are declared to be "one way" only. Alice has no information on the direction of the roads, but the King of Hearts has offered to help her. She is allowed to ask him a number of questions. For each question in turn, Alice chooses a pair of towns and the King of Hearts tells her the direction of the road connecting those two towns.

   Alice wants to know whether there is at least one town in Wonderland with at most one outgoing road. Prove that she can always find out by asking at most $4n$ questions.

3. Let a be a positive integer. We say that a positive integer b is *a-good* if $\binom{an}{b} - 1$ is divisible by $an + 1$ for all positive integers n with $an \geq b$. Suppose b is a positive integer such that b is a-good, but $b + 2$ is not a-good. Prove that $b + 1$ is prime.

Thailand Team Selection Test for IMO 2020
IPST, Bangkok
22 January 2020

TST 1 Day 2
Time: 4.5 hours

4. Let r be a positive integer. Find all positive integers k such that there exist an odd integer $m \geqslant 3$ and a positive integer n satisfying

$$k \mid m^{2^r} - 1 \quad \text{and} \quad m \mid n^{\frac{1}{k}(m^{2^r}-1)} + 1.$$

5. Let x, y, z be nonnegative real numbers such that $x + y + z = 3$. Prove that

$$\frac{x}{4-y} + \frac{y}{4-z} + \frac{z}{4-x} + \frac{1}{16}(1-x)^2(1-y)^2(1-z)^2 \leqslant 1,$$

and determine all such triples (x, y, z) where the equality holds.

6. Let $n \geqslant 2$ be an integer. Suppose $2n$ points are given in a plane such that no three of them are collinear. The points are to be labeled $A_1, A_2, \ldots, A_{2n}$ in some order. Consider the $2n$ angles

$$\angle A_1A_2A_3, \angle A_2A_3A_4, \ldots, \angle A_{2n-2}A_{2n-1}A_{2n}, \angle A_{2n-1}A_{2n}A_1, \angle A_{2n}A_1A_2.$$

Each angle is measured in the way that gives the smallest positive value (i.e. between $0°$ and $180°$). Prove that there exists an ordering of the given points such that the resulting $2n$ angles can be separated into two groups with the sum of one group of angles equal to the sum of the other group.

Thailand Team Selection Test for IMO 2020
IPST, Bangkok
30 January 2020

TST 2 Day 1
Time: 4.5 hours

1. A set of n blocks is given, each weighing at least 1; their total weight is $2n$. Prove that for every real number r with $0 \leqslant r \leqslant 2n-2$ you can choose a subset of the blocks whose total weight is at least r but at most $r+2$.

2. Let P be a point inside triangle ABC. Let AP meet BC at A_1, let BP meet CA at B_1, and let CP meet AB at C_1. Let A_2 be the point such that A_1 is the midpoint of PA_2, let B_2 be the point such that B_1 is the midpoint of PB_2, and let C_2 be the point such that C_1 is the midpoint of PC_2. Prove that points A_2, B_2, and C_2 cannot all lie strictly inside the circumcircle of triangle ABC.

3. Let $x_1, x_2, \ldots, x_n$ be different real numbers. Prove that

$$\sum_{1\leqslant i\leqslant n} \prod_{j\neq i} \frac{1-x_i x_j}{x_i - x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Thailand Team Selection Test for IMO 2020
IPST, Bangkok
31 January 2020

TST 2 Day 2
Time: 4.5 hours

4. Let n be a positive integer and let P be the set of monic polynomials of degree n with complex coefficients. Find the value of
$$\min_{p \in P} \left\{ \max_{|z|=1} |p(z)| \right\}.$$

5. A positive integer C is given. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $a + f(b)$ divides $a^2 + bf(a)$ for all positive integers a and b with $a + b > C$.

6. Let I be the incenter of acute triangle ABC. Let the incircle meet BC, CA, and AB at D, E, and F, respectively. Let line EF intersect the circumcircle of the triangle at P and Q, such that F lies between E and P. Prove that $\angle DPA + \angle AQD = \angle QIP$.

Thailand Team Selection Test for IMO 2020
IPST, Bangkok
19 March 2020

TST 3 Day 1
Time: 4.5 hours

1. The infinite sequence $a_0, a_1, a_2, \ldots$ of (not necessarily distinct) integers has the following properties: $0 \leqslant a_i \leqslant i$ for all integers $i \geqslant 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \geqslant 0$.

Prove that all integers $N \geqslant 0$ occur in the sequence (that is, for all $N \geqslant 0$, there exists $i \geqslant 0$ with $a_i = N$).

2. Let $ABCDE$ be a convex pentagon such that $\angle EDC \neq 2 \cdot \angle ADB$ and $CD = DE$. Suppose P is an interior point of the pentagon such that $AP = AE$ and $BP = BC$. Prove that P lies on the diagonal CE if and only if

$$\text{area}(BCD) + \text{area}(ADE) = \text{area}(ABD) + \text{area}(ABP).$$

3. Let a and b be positive integers. Prove that

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a perfect square. (Here $\lceil x \rceil$ denotes the least integer greater than or equal to x.)

Thailand Team Selection Test for IMO 2020
IPST, Bangkok
20 March 2020

TST 3 Day 2
Time: 4.5 hours

4. Find all triples (a, b, c) of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.

5. Let $n \geqslant 2$ be a positive integer and $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = 0$. Define the set A by

$$A = \{(i, j) \mid 1 \leqslant i < j \leqslant n, |a_i - a_j| \geqslant 1\}$$

Prove that, if A is not empty, then

$$\sum_{(i,j) \in A} a_i a_j < 0.$$

6. There are $N \geqslant 2$ empty boxes $B_1, B_2, \ldots, B_N$ in a row on a table and an unlimited supply of pebbles. Given a positive integer n, Pic and Ping play the following *game of darkness*.

   In the first round, Pic takes n pebbles and distributes them into the N boxes as he wishes. Each subsequent round consists of two steps:

   (i) Ping chooses an integer k with $1 \leqslant k \leqslant N - 1$ and splits the boxes into two groups $B_1, B_2, \ldots, B_k$ and $B_{k+1}, B_{k+2}, \ldots, B_N$.

   (ii) Pic picks one of these two groups, add one pebble to each box in that group, and removes one pebble from each box in the other group.

   Ping will be the next team leader if there is a positive integer i such that, at the end of the i^{th} round, there is a box containing no pebbles. Find the smallest n such that Pic can prevent Ping from becoming the next team leader.

Thailand Team Selection Test for IMO 2020
IPST, Bangkok
2 May 2020

TST 4 Day 1
Time: 4.5 hours

1. Let ABC be a triangle with circumcircle Γ. Let ω_0 be a circle tangent to chord AB and arc ACB. For each $i = 1, 2$, let ω_i be a circle tangent to AB at T_i, to ω_0 at S_i, and to arc ACB.

   Suppose $\omega_1 \neq \omega_2$. Prove that there is a circle passing through S_1, S_2, T_1, and T_2, and tangent to Γ if and only if $\angle ACB = 90°$.

2. On a flat plane in Camelot, King Arthur builds a labyrinth consisting of n walls, each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.

   At each intersection of any two walls, there is a two-way door connecting the two diagonally opposite corners at which sides of different colors meet. Morgana then places some knights in the labyrinth. The knights can walk through doors, but cannot walk through walls.

   Find the largest k such that, regardless of how the labyrinth is built or how the walls are painted, Morgana can always place k knights such that no two of them can ever meet.

3. Consider functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\left(f(x+y)+y\right) = f\left(f(x)+y\right)$$

   for all integers x and y. An integer v is said to be f-*rare* if the set $X_v = \{x \in \mathbb{Z} \mid f(x) = v\}$ is finite and nonempty. Prove that no such function f can have more than one f-rare integer.

Thailand Team Selection Test for IMO 2020
IPST, Bangkok
3 May 2020

TST 4 Day 2
Time: 4.5 hours

4. Let $N \geqslant 2$ be an integer and $u_1, u_2, \ldots, u_{2019}$ be real numbers satisfying

$$u_1 + u_2 + \cdots + u_N = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \cdots + u_N^2 = 1.$$

   Let $a = \min\{u_1, u_2, \ldots, u_N\}$ and $b = \max\{u_1, u_2, \ldots, u_N\}$. Prove that $ab \leqslant -\frac{1}{N}$.

5. A set S of integers is said to be *rootiful* if, for any positive integer n and any integers $a_0, a_1, \cdots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1x + \cdots + a_nx^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers $a \geqslant b$.

6. Let $\mathcal{L}$ be the set of all lines in the plane and let f be a function that assigns to each line $\ell \in \mathcal{L}$ a point $f(\ell)$ on ℓ. Suppose that for any point X, and for any three lines ℓ_1, ℓ_2, ℓ_3 passing through X, the points $f(\ell_1), f(\ell_2), f(\ell_3)$, and X lie on a circle. Prove that there is a unique point P such that $f(\ell) = P$ for any line ℓ passing through P.