## Day 1 Time: 4.5 hours

1. Let x, y, z be positive reals such that xyz = 1. Prove that

$$\sum_{\text{cyc}} \frac{1}{\sqrt{x+2y+6}} \leqslant \sum_{\text{cyc}} \frac{x}{\sqrt{x^2+4\sqrt{y}+4\sqrt{z}}}.$$

- 2. A positive integer n < 2017 is given. Exactly *n* vertices of a regular 2017-gon are colored red, and the remaining vertices are colored blue. Prove that the number of isosceles triangles whose vertices are monochromatic does not depend on the chosen coloring (but does depend on *n*.)
- 3. Does there exist an arithmetic progression with 2017 terms such that each term is not a perfect power, but the product of all 2017 terms is?
- 4. Let  $\triangle ABC$  be an acute triangle with altitudes  $AA_1, BB_1, CC_1$  and orthocenter H. Let K, L be the midpoints of  $BC_1, CB_1$ . Let  $\ell_A$  be the external angle bisector of  $\angle BAC$ . Let  $\ell_B, \ell_C$  be the lines through B, C perpendicular to  $\ell_A$ . Let  $\ell_H$  be the line through H parallel to  $\ell_A$ . Prove that the centers of the circumcircles of  $\triangle A_1B_1C_1$ ,  $\triangle AKL$  and the rectangle formed by  $\ell_A, \ell_B, \ell_C, \ell_H$  lie on the same line.



# Thailand Team Selection Test for IMO 2018 IPST, Bangkok 29 January 2018

#### Day 2 Time: 4.5 hours

1. Determine all integers  $n \ge 2$  having the following property: for any integers  $a_1, a_2, \ldots, a_n$  whose sum is not divisible by n, there exists an index  $1 \le i \le n$  such that none of the numbers

 $a_i, a_i + a_{i+1}, \dots, a_i + a_{i+1} + \dots + a_{i+n-1}$ 

is divisible by n. Here, we let  $a_i = a_{i-n}$  when i > n.

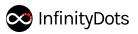
2. For finite sets A, M such that  $A \subseteq M \subset \mathbb{Z}^+$ , we define

 $f_M(A) = \{x \in M \mid x \text{ is divisible by an odd number of elements of } A\}.$ 

Given a positive integer k, we call M k-colorable if it is possible to color the subsets of M with k colors so that for any  $A \subseteq M$ , if  $f_M(A) \neq A$  then  $f_M(A)$  and A have different colors.

Determine the least positive integer k such that every finite set  $M \subset \mathbb{Z}^+$  is k-colorable.

3. Let S be a finite set, and let  $\mathcal{A}$  be the set of all functions from S to S. Let f be an element of  $\mathcal{A}$ , and let T = f(S) be the image of S under f. Suppose that  $f \circ g \circ f \neq g \circ f \circ g$  for every g in  $\mathcal{A}$  with  $g \neq f$ . Show that f(T) = T.



# Thailand Team Selection Test for IMO 2018 IPST, Bangkok 30 January 2018

### Day 3 Time: 4.5 hours

- 1. A rectangle  $\mathcal{R}$  with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of  $\mathcal{R}$  are either all odd or all even.
- 2. Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC
- 3. Find the smallest positive integer n or show no such n exists, with the following property: there are infinitely many distinct n-tuples of positive rational numbers  $(a_1, a_2, \ldots, a_n)$  such that both

$$a_1 + a_2 + \dots + a_n$$
 and  $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$ 

are integers.





# Thailand Team Selection Test for IMO 2018 IPST, Bangkok 11 March 2018

#### Day 4 Time: 4.5 hours

1. Let  $a_1, a_2, \ldots, a_n, k$ , and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = k$$
 and  $a_1 a_2 \dots a_n = M$ .

If M > 1, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

2. Let  $(x_1, x_2, ..., x_{100})$  be a permutation of (1, 2, ..., 100). Define

 $S = \{m \mid m \text{ is the median of } \{x_i, x_{i+1}, x_{i+2}\} \text{ for some } i\}.$ 

Determine the minimum possible value of the sum of all elements of S.

3. Let  $ABCC_1B_1A_1$  be a convex hexagon such that AB = BC, and suppose that the line segments  $AA_1, BB_1$ , and  $CC_1$  have the same perpendicular bisector. Let the diagonals  $AC_1$  and  $A_1C$  meet at D, and denote by  $\omega$  the circle ABC. Let  $\omega$  intersect the circle  $A_1BC_1$  again at  $E \neq B$ . Prove that the lines  $BB_1$  and DE intersect on  $\omega$ .



# Thailand Team Selection Test for IMO 2018 IPST, Bangkok 19 March 2018

## Day 5 Time: 4.5 hours

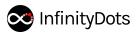
- 1. Let ABCDE be a convex pentagon such that AB = BC = CD,  $\angle EAB = \angle BCD$ , and  $\angle EDC = \angle CBA$ . Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.
- 2. Find all pairs (p,q) of prime numbers which p > q and

$$\frac{(p+q)^{p+q}(p-q)^{p-q}-1}{(p+q)^{p-q}(p-q)^{p+q}-1}$$

is an integer.

3. Let n > 1 be a given integer. An  $n \times n \times n$  cube is composed of  $n^3$  unit cubes. Each unit cube is painted with one colour. For each  $n \times n \times 1$  box consisting of  $n^2$  unit cubes (in any of the three possible orientations), we consider the set of colours present in that box (each colour is listed only once). This way, we get 3n sets of colours, split into three groups according to the orientation.

It happens that for every set in any group, the same set appears in both of the other groups. Determine, in terms of n, the maximal possible number of colours that are present.



## Day 6 Time: 4.5 hours

- 1. Let *E* and *F* be points on side *BC* of a triangle  $\triangle ABC$ . Points *K* and *L* are chosen on segments *AB* and *AC*, respectively, so that *EK*  $\parallel$  *AC* and *FL*  $\parallel$  *AB*. The incircles of  $\triangle BEK$  and  $\triangle CFL$  touches segments *AB* and *AC* at *X* and *Y*, respectively. Lines *AC* and *EX* intersect at *M*, and lines *AB* and *FY* intersect at *N*. Given that *AX* = *AU*, prove that  $MN \parallel BC$ .
- 2. Call a rational number *short* if it has finitely many digits in its decimal expansion. For a positive integer m, we say that a positive integer t is m-tastic if there exists a number  $c \in \{1, 2, 3, \ldots, 2017\}$  such that  $\frac{10^t-1}{c \cdot m}$  is short, and such that  $\frac{10^k-1}{c \cdot m}$  is not short for any  $1 \leq k < t$ .

Let S(m) be the set of *m*-tastic numbers. Consider S(m) for m = 1, 2, ... What is the maximum number of elements in S(m)?

3. Let  $n \ge 3$  be an integer. Let  $a_1, a_2, \ldots, a_n \in [0, 1]$  satisfy  $a_1 + a_2 + \cdots + a_n = 2$ . Prove that

$$\sqrt{1-\sqrt{a_1}} + \sqrt{1-\sqrt{a_2}} + \dots + \sqrt{1-\sqrt{a_n}} \le n-3 + \sqrt{9-3\sqrt{6}}.$$



## Day 7 Time: 4.5 hours

1. Determine all functions  $g : \mathbb{R} \to \mathbb{R}$  for which there exists a strictly monotone function  $f : \mathbb{R} \to \mathbb{R}$  such that

$$f(x+y) = f(x)g(y) + f(y)$$

for all reals x, y.

- 2. Sir Alex plays the following game on a row of 9 cells. Initially, all cells are empty. In each move, Sir Alex is allowed to perform exactly one of the following two operations:
  - (i) Choose any number of the form  $2^j$ , where j is a non-negative integer, and put it into an empty cell.
  - (ii) Choose two (not necessarily adjacent) cells with the same number in them; denote that number by  $2^{j}$ . Replace the number in one of the cells with  $2^{j+1}$  and erase the number in the other cell.

At the end of the game, one cell contains  $2^n$ , where n is a given positive integer, while the other cells are empty. Determine the maximum number of moves that Sir Alex could have made, in terms of n.

3. Let n be a fixed odd positive integer. For each odd prime p, define

$$a_p = \frac{1}{p-1} \sum_{k=1}^{\frac{p-1}{2}} \left\{ \frac{k^{2n}}{p} \right\}.$$

Prove that there is a real number c such that  $a_p = c$  for infinitely many primes p. Note:  $\{x\} = x - \lfloor x \rfloor$  is the fractional part of x.



# Thailand Team Selection Test for IMO 2018 IPST, Bangkok 29 April 2018

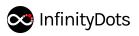
#### Day 8 Time: 4.5 hours

- 1. Let n be a positive integer. Define a chameleon to be any sequence of 3n letters, with exactly n occurrences of each of the letters a, b, and c. Define a swap to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X, there exists a chameleon Y such that X cannot be changed to Y using fewer than  $3n^2/2$  swaps.
- 2. A sequence of real numbers  $a_1, a_2, \ldots$  satisfies the relation

$$a_n = -\max_{i+j=n} (a_i + a_j) \quad \text{for all} \quad n > 2017.$$

Prove that the sequence is bounded, i.e., there is a constant M such that  $|a_n| \leq M$  for all positive integers n.

3. A convex quadrilateral ABCD has an inscribed circle with center I. Let  $I_a, I_b, I_c$  and  $I_d$  be the incenters of the triangles DAB, ABC, BCD and CDA, respectively. Suppose that the common external tangents of the circles  $AI_bI_d$  and  $CI_bI_d$  meet at X, and the common external tangents of the circles  $BI_aI_c$  and  $DI_aI_c$  meet at Y. Prove that  $\angle XIY = 90^{\circ}$ .



# Thailand Team Selection Test for IMO 2018 IPST, Bangkok 30 April 2018

#### Day 9 Time: 4.5 hours

1. Let  $p \ge 2$  be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index i in the set  $\{1, 2, \ldots, p-1\}$  that was not chosen before by either of the two players and then chooses an element  $a_i$  from the set  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ . Eduardo has the first move. The game ends after all the indices have been chosen. Then the following number is computed:

$$M = a_0 + a_1 10 + a_2 10^2 + \dots + a_{p-1} 10^{p-1} = \sum_{i=0}^{p-1} a_i 10^i.$$

The goal of Eduardo is to make M divisible by p, and the goal of Fernando is to prevent this. Prove that Eduardo has a winning strategy.

- 2. In triangle ABC, let  $\omega$  be the excircle opposite to A. Let D, E and F be the points where  $\omega$  is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to  $\omega$ .
- 3. An integer  $n \ge 3$  is given. We call an *n*-tuple of real numbers  $(x_1, x_2, \ldots, x_n)$  Shiny if for each permutation  $y_1, y_2, \ldots, y_n$  of these numbers, we have

$$\sum_{i=1}^{n-1} y_i y_{i+1} = y_1 y_2 + y_2 y_3 + y_3 y_4 + \dots + y_{n-1} y_n \ge -1.$$

Find the largest constant K = K(n) such that

$$\sum_{1 \leqslant i < j \leqslant n} x_i x_j \geqslant K$$

holds for every Shiny *n*-tuple  $(x_1, x_2, \ldots, x_n)$ .

