Thailand Team Selection Test for IMO 2017
IPST, Bangkok
22 December 2016

Day 1
Time: 4.5 hours

1. An *altitude* of a convex pentagon is a line through a vertex that is perpendicular to the opposite side. Show that if four altitudes of a convex pentagon intersect at a single point, then the fifth altitude also passes through that point.

2. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation
$$f(f(x) + yz) = x + f(y)f(z)$$
for all real numbers x, y, z.

3. Let $n > 1$ be an integer. The leader of an IMO team picks an n-digit binary string and tells it secretly to the deputy leader. The deputy leader then picks a positive integer $k < n$ in secret, and writes down all n-digit binary strings which differ from the leader?s in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down $001, 111$ and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader?s string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

4. For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geqslant 2016$, the integer $P(n)$ is positive and
$$S(P(n)) = P(S(n)).$$

Thailand Team Selection Test for IMO 2017
IPST, Bangkok
24 January 2017

Day 2
Time: 4.5 hours

1. Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geqslant 1$. Prove that
$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leqslant \left(\frac{a+b+c}{3}\right)^2 + 1.$$

2. Two circles Γ_1, Γ_2 intersect at M, N. A line ℓ cuts Γ_1 at A, C and Γ_2 at B, D such that A, B, C, D are all distinct and lie in this order. A point X is chosen on line MN so that M lies between X, N. Lines AX and BM intersect at P, and lines DX and CM intersect at Q. If K is the midpoint of AD and L is the midpoint of BC then prove that KX and LM intersect at a point on line PQ.

3. Let n be a positive integer; n boys and n girls stand in a circle. A pair consisting of a boy and a girl is said to be a *balancing pair* if there are an equal number (possibly zero) of boys and girls in both arcs between the pair. If there is a girl who is in exactly 10 balancing pairs, prove that there is also a boy who is in exactly 10 balancing pairs.

4. Let m, n, k, ℓ be positive integers with $n \neq 1$ such that $n^k + mn^\ell + 1$ divides $n^{k+\ell} - 1$. Prove that either
   - $m = 1$ and $\ell = 2k$, or
   - $\ell \mid k$ and $m = \frac{n^{k-\ell}-1}{n^\ell - 1}$.

Thailand Team Selection Test for IMO 2017
IPST, Bangkok
25 January 2017

Day 3
Time: 4.5 hours

1. Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints:
   - each cell contains a distinct divisor;
   - the sums of all rows are equal; and
   - the sums of all columns are equal.

2. Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

3. Find all positive integers n such that the following statement holds: Suppose real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ satisfy $|a_k| + |b_k| = 1$ for all $k = 1, \ldots, n$. Then there exists $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, each of which is either -1 or 1, such that
$$\left|\sum_{i=1}^{n} \varepsilon_i a_i\right| + \left|\sum_{i=1}^{n} \varepsilon_i b_i\right| \leqslant 1.$$

Thailand Team Selection Test for IMO 2017
IPST, Bangkok
7 March 2017

Day 4
Time: 4.5 hours

1. Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all positive integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

2. Find all functions $f : (0, \infty) \to (0, \infty)$ such that for any $x, y \in (0, \infty)$,

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right).$$

3. Let A_1, B_1 and C_1 be points on sides BC, CA and AB of an acute triangle ABC respectively, such that AA_1, BB_1 and CC_1 are the internal angle bisectors of triangle ABC. Let I be the incentre of triangle ABC, and H be the orthocentre of triangle $A_1B_1C_1$. Show that

$$AH + BH + CH \geq AI + BI + CI.$$

Thailand Team Selection Test for IMO 2017
IPST, Bangkok
10 March 2017

Day 5
Time: 4.5 hours

1. Let $n \geqslant 2$ be an integer. *Killer* is a game played by a dealer and n players. The game begins with the dealer designating one of the n players a killer and keeping this information a secret. Every player knows that the killer exists and is among the n players. The dealer can make as many public announcements as he wishes. Then, he secretly gives each of the n players a (possibly different) name of one of the n players. This game has the property that:

   (i) Alone, each player (killer included) does not know who is the killer. Each player also cannot tell with certainty who is not the killer.

   (ii) If any 2 of the n players exchange information, they can determine the killer.

   For example, if there are a dealer and 2 players, the dealer can announce that he will give the same name to both players if the first player is the killer, and give different names to each player if the second player is the killer.

   a) Prove that Killer can be played with a dealer and 5 players.

   b) Determine whether Killer can be played with a dealer and 4 players.

2. Let $\triangle ABC$ be a triangle with $\angle A = 60^\circ$ and $AB > AC$. Let O be its circumcenter, F the foot of the altitude from C, and D a point on the side AB such that $BD = AC$. Suppose that the points O, F, and D are distinct. Prove that the circumcircle of $\triangle OFD$ intersects the circle centered at O with radius OF on the altitude of $\triangle ABC$ from B.

3. Genji the ninja is to jump along the real axis, starting at the point 0. In doing so, he alternates between the following two types of jumps.

   Type 1: Genji jumps from the current position x to a point y in the set

   $$\{x, x+1, x+2, x+3, x+4, x+5, x+6\}.$$

   Type 2: Genji jumps from the current position y to $(\sqrt{2}-3)y$.

   Let a and b be integers. Prove that there exists a finite sequence of jumps that allows Genji to land on the point $a + b\sqrt{2}$.

Thailand Team Selection Test for IMO 2017
IPST, Bangkok
21 March 2017

Day 6
Time: 4.5 hours

1. Let $(2+\sqrt{3})^k = m + n\sqrt{3}$, where m, n, k are positive integers and k is odd. Prove that $\sqrt{m-1}$ is an integer.

2. Prove that $a = \frac{4}{9}$ is the largest real number such that the inequality
$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \geqslant a\left(\frac{2}{x_1} + \frac{3}{x_2} + \cdots + \frac{n+1}{x_n}\right)$$
holds for all $n \geqslant 1$ and for all real numbers $x_0, x_1, ..., x_n$ satisfying $0 = x_0 < x_1 < x_2 < \cdots < x_n$ we have

3. There are $n \geqslant 3$ islands in a city. Initially, the ferry company offers some routes between some pairs of islands so that it is impossible to divide the islands into two groups such that no two islands in different groups are connected by a ferry route.

   After each year, the ferry company will close a ferry route between some two islands X and Y. At the same time, in order to maintain its service, the company will open new routes according to the following rule: for any island which is connected to a ferry route to exactly one of X and Y, a new route between this island and the other of X and Y is added.

   Suppose at any moment, if we partition all islands into two nonempty groups in any way, then it is known that the ferry company will close a certain route connecting two islands from the two groups after some years. Prove that after some years there will be an island which is connected to all other islands by ferry routes.

Thailand Team Selection Test for IMO 2017
IPST, Bangkok
25 March 2017

Day 7
Time: 4.5 hours

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \neq 0$ and
$$f(x+y)^2 = 2f(x)f(y) + \max\left\{f(x^2+y^2), f(x^2)+f(y^2)\right\}$$
for all real numbers x and y.

2. Find all polynomials $P(x)$ of odd degree d and with integer coefficients with the following property: for each positive integer n, there exists n positive integers $x_1, x_2, \ldots, x_n$ such that
$$\frac{P(x_i)}{P(x_j)}$$
lies in the interval $(\frac{1}{2}, 2)$ and is a d^{th} power of a rational number for every pair of indices i and j with $1 \leq i, j \leq n$.

Thailand Team Selection Test for IMO 2017
IPST, Bangkok
26 March 2017

Day 8
Time: 4.5 hours

1. Let $n \geq 3$ be an integer. Each edge of a complete graph with n vertices is colored in one of three given colors so that each color is used at least once. Determine the smallest positive integer k such that it is always possible to recolor k edges in the graph so that all vertices of the graph are connected by edges of a single color.

2. Let I be the incenter of a non-equilateral triangle ABC, I_A be the A-excenter, I'_A be the reflection of I_A in BC, and ℓ_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line ℓ_B analogously. Let P be the intersection point of ℓ_A and ℓ_B.

   a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

   b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120°$.

Thailand Team Selection Test for IMO 2017
IPST, Bangkok
2 April 2017

Day 9
Time: 4.5 hours

1. Alice and Bob plays a game as follows.

   Step 1. Alice draws some n distinct circles in the plane, creating several bounded regions.

   Step 2. Bob color each bounded region red or blue.

   Step 3. Alice may choose to change the color of a single region.

   Alice wins if there is an even number of red regions in each circle, else Bob wins. Determine all n such that Alice has a winning strategy.

2. Let $\{x_n\}$ be a sequence satisfying $x_1 = 2$ and

$$x_n = 4\nu_3(n) + 2 - \frac{2}{x_{n-1}}$$

   for all $n \geq 2$. Prove that each positive rational number appears exactly once in this sequence.

3. a) Prove that for all positive integers n, there exists integers a, b satisfying $0 < b \leqslant \sqrt{n}+1$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

   b) Show that there are infinitely many positive integers n for which there are no integers a, b satisfying $0 < b \leqslant \sqrt{n}$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

Thailand Team Selection Test for IMO 2017
IPST, Bangkok
3 April 2017

Day 10
Time: 4.5 hours

1. Show that there are infinitely many pairs of distinct prime numbers (m, n) such that

$$3^n \equiv 3 \pmod{m} \quad \text{and} \quad 3^m \equiv 3 \pmod{n}.$$

2. Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be *nice* if

   (i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

   (ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

   Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) : x \geqslant 0, y \geqslant 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

3. Let $n \geqslant 3$ be a positive integer. Find the maximum number of diagonals in a regular n-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

Thailand Team Selection Test for IMO 2017
IPST, Bangkok
11 April 2017

Day 11
Time: 4.5 hours

1. Let P be a point inside $\triangle ABC$. Points A_1, B_1, and C_1 are chosen on segments PA, PB, and PC respectively. Let $BC_1 \cap CB_1 = \{A_2\}$, $CA_1 \cap AC_1 = \{B_2\}$, and $AB_1 \cap BA_1 = \{C_2\}$. Let U be the intersection of the lines A_1B_1 and A_2B_2 and let V be the intersection of the lines A_1C_1 and A_2C_2. Show that the lines UC_2,VB_2, and AP are concurrent.

2. Let X be a point inside a convex 100-gon that does not lie on any side or diagonal of the 100-gon. Pete and Basil play a game as follows: first, Pete marks *two* vertices of the 100-gon then they alternately mark a vertex of the 100-gon, and the player who makes a move so that X lies inside the polygon formed by the marked vertices loses. Prove that Pete has a winning strategy.

3. Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

Thailand Team Selection Test for IMO 2017
IPST, Bangkok
12 April 2017

Day 12
Time: 4.5 hours

1. Prove that for all positive real numbers a, b, c,

$$\frac{a}{\sqrt{3ab+bc}}+\frac{b}{\sqrt{3bc+ca}}+\frac{c}{\sqrt{3ca+ab}} \geqslant \frac{3}{2}.$$

2. Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcenter and the orthocenter) of an acute scalene triangle ABC. A circle ω with center S passes through A and D, and intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcenter of triangle XSY is equidistant from P and M.

3. Let H be a set whose elements are finite sequences of numbers from $1, 2, 3$. Suppose that no element of H is a subsequence of any other element of H. Show that H is finite.