Thailand Team Selection Test for IMO 2016
IPST, Bangkok
25 December 2015

Day 1
Time: 3 hours

1. Santa has k piles of gifts, containing $h_1, h_2, \ldots, h_k$ gifts each. If

   (i) $1 \leqslant h_1 < h_2 < \cdots < h_k$, and

   (ii) no matter how Santa split a pile into two piles, there will always be two piles in the resulting $k+1$ piles that contain equal number of gifts.

   Prove that for all $1 \leqslant s \leqslant k$, $h_s \leqslant 2s$.

2. Let $\triangle ABC$ be an acute-angled triangle whose altitudes AA_1 and BB_1 intersect at H. Let ω_1 be the circle centered at H passing through B_1 and let ω_2 be the circle centered at B passing through B_1. Let CN and CK be the tangent lines from C to circles ω_1 and ω_2 respectively (N and K are distinct from B_1). Prove that A_1, N and K are collinear.

3. Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$.

4. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

   holds for all $x, y \in \mathbb{Z}$.

Thailand Team Selection Test for IMO 2016
IPST, Bangkok
14 January 2016

Day 2
Time: 4.5 hours

1. Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \leqslant r < s \leqslant 2n} (s - r - n) x_r x_s,$$

where $-1 \leqslant x_i \leqslant 1$ for all $i = 1, \ldots, 2n$.

2. Each square of $n \times n$ is colored by one of n different colors so that there are n unit squares of each color. Prove that there is a row or column containing at least $\sqrt{n}$ distinct colors.

3. Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

Thailand Team Selection Test for IMO 2016
IPST, Bangkok
15 January 2016

Day 3
Time: 4.5 hours

1. Let ω be the circumcircle of triangle ABC, and let M be a midpoint of $\overset{\frown}{BC}$, not containing A. The incircle of $\triangle ABC$ is centered at I and touches BC at D. The A-excircle of $\triangle ABC$ is centered at I_A and touches BC at E. Lines MD and ME are drawn intersecting ω again at points $T \neq D$ and $R \neq E$, respectively. Let RI_A intersect ω at $S \neq R$. Show that T, I, and S are collinear.

2. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that
$$\frac{\sqrt{2}a^2b}{2a+b} + \frac{\sqrt{2}b^2c}{2b+c} + \frac{\sqrt{2}c^2a}{2c+a} \leq \frac{\sqrt{a^2+b^2}}{2ab+1} + \frac{\sqrt{b^2+c^2}}{2bc+1} + \frac{\sqrt{c^2+a^2}}{2ca+1}$$

3. For a finite set A of positive integers, a partition of A into two disjoint nonempty subsets A_1 and A_2 is *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

Thailand Team Selection Test for IMO 2016
IPST, Bangkok
14 March 2016

Day 4
Time: 4.5 hours

1. Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segment PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

2. Let n be a fixed integer with $n \geqslant 2$. We say that two polynomials P and Q with real coefficients are *twins* if for each $i \in \{1, 2, \ldots, n\}$ the sequences

$$P(2015i), P(2015i-1), \ldots, P(2015i-2014) \quad \text{and}$$
$$Q(2015i), Q(2015i-1), \ldots, Q(2015i-2014)$$

are permutations of each other.

a) Prove that there exist distinct twin polynomials of degree $n+1$.

b) Prove that there do not exist distinct twin polynomials of degree n.

Thailand Team Selection Test for IMO 2016
IPST, Bangkok
16 March 2016

Day 5
Time: 4.5 hours

1. A set of $n \geqslant 5$ points on the plane, no three of which are collinear, is given. Hillary and Donald play a game as follows: the players will alternately choose a pair of points not yet joined by a segment, and draw a segment connecting them. If after a player's turn, each of the n points is an endpoint of at least one drawn segment, then that player wins. If Hillary goes first, then for which values of n will Donald win regardless of Hillary's strategy?

2. Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of BT/BM.

3. Suppose that $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ are two sequences of positive integers such that $a_0, b_0 \geqslant 2$ and
$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1.$$
Show that the sequence a_n is eventually periodic; in other words, there exist integers $N \geqslant 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geqslant N$.

Thailand Team Selection Test for IMO 2016
IPST, Bangkok
25 March 2016

Day 6
Time: 4.5 hours

1. Find a monic polynomial $P(x)$ with rational coefficients such that $P(\sqrt[3]{3}) = \sqrt[3]{2}$ having the minimum possible degree, or prove that no such polynomial exists.

2. Dumbledore has n bags of candies, containing a total of n^2 candies. Dumbledore can choose two bags containing a total of an even number of candies, and transfer candies between them so that there are equally many candies in each. Find all integers $n > 2$ such that Dumbledore can always make all bags contain an equal amount of candies no matter how the candies are distributed in the beginning.

3. Two circles ω_1 and ω_2 intersect at points A and B. A line ℓ_1 through A intersects ω_1 and ω_2 again at C and E, respectively, and a point G is chosen on this line between A and E. A line ℓ_2 through B intersects ω_1 and ω_2 again at D and F, respectively, and a point H is chosen on this line between B and F. Line CH intersects FG at I and intersects ω_1 again at J. Line DG intersects EH at K and intersects ω_1 again at L. Lines EH and FG intersect ω_2 again at M and N, respectively. Assume that points $A, B, \ldots, N$ are all distinct. Prove that I, J, K, L, M, N lie on a circle.

Thailand Team Selection Test for IMO 2016
IPST, Bangkok
26 March 2016

Day 7
Time: 4.5 hours

1. Let ABC be a triangle with circumcircle ω. Let D be a point on AB. Let Γ be the circle which is tangent to line DB at M, line DC at N, and also to ω externally. Suppose that the external angle bisector of $\angle ABC$ intersects MN at X. Show that AX bisects $\angle BAC$.

2. Let $a_1 = 11^{11}, a_2 = 12^{12}, a_3 = 13^{13}$, and

$$a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|$$

for all $n \geq 4$. Determine $a_{14^{14}}$.

3. Let $2\mathbb{Z} + 1$ denote the set of odd integers. Find all functions $f : \mathbb{Z} \mapsto 2\mathbb{Z} + 1$ satisfying

$$f(x + f(x) + y) + f(x - f(x) - y) = f(x + y) + f(x - y)$$

for every $x, y \in \mathbb{Z}$.

Thailand Team Selection Test for IMO 2016
IPST, Bangkok
20 April 2016

Day 8 Morning
Time: 2 hours

1. Find all complex numbers $z = a + bi$ with $a, b \in \mathbb{Z}$ satisfying the equation

$$3|z|^2 + |z + 4 + 2i|^2 + 6(z + i - z)^2 = 14.$$

2. Find all continuous periodic functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$(1 + f(x)f(y))f(x + y) = f(x) + f(y)$$

for all real numbers x and y.

3. Let n be an integer such that $n \geq 2$. Color each square of an $n \times n$ table either red or blue. A domino is called *chic* if it covers squares of different colors, and *cool* if it covers two squares of the same color.

Find the largest integer k such that no matter how the table is colored, we can always put k non-overlapping dominoes on the table such that they are either all chic or all cool.

Thailand Team Selection Test for IMO 2016
IPST, Bangkok
20 April 2016

Day 8 Afternoon
Time: 2 hours

1. Given a 2015-tuple $(a_1, a_2, \ldots, a_{2015})$, we are allowed to choose $1 \le m, n \le 2015$ such that a_m is even, and replace a_m and a_n with $\frac{a_m}{2}$ and $a_n + \frac{a_m}{2}$ respectively.

   Prove that starting with $(1, 2, \ldots, 2015)$, we can arrive at any permutation of $(1, 2, \ldots, 2015)$ by a finite sequence of moves.

2. Let $\triangle ABC$ be a triangle with circumcircle ω. The circle centered at B having radius BC intersects AC and ω at D and E, respectively. The line through D parallel to CE intersects AB at F.

   a) Prove that BD and EF intersect on ω

   b) Let G be the intersection point of AB and CE. Prove that $\square DEFG$ is a rhombus.

3. Prove that for any prime number p and positive integer k, there exists a positive integer n such that the decimal representation of p^n contains a string of k consecutive digits, all of which are equal.

Thailand Team Selection Test for IMO 2016
IPST, Bangkok
21 April 2016

Day 9
Time: 4.5 hours

1. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$. Suppose that f has the following two properties:

   (i) if $m, n \in \mathbb{Z}_{>0}$, then $\frac{f^n(m)-m}{n} \in \mathbb{Z}_{>0}$;

   (ii) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

   Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

2. Let S be a nonempty set of positive integers. We say that a positive integer n is *clean* if it has a unique representation as a sum of an odd number of distinct elements from S. Prove that there exist infinitely many positive integers that are not clean.

Thailand Team Selection Test for IMO 2016
IPST, Bangkok
28 April 2016

Day 10
Time: 4.5 hours

1. In Lineland there are $n \geqslant 1$ towns, arranged along a road running from left to right. Each town has a *left bulldozer* (put to the left of the town and facing left) and a *right bulldozer* (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

   Let A and B be two towns, with B to the right of A. We say that town A can *sweep* town B *away* if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly town B can sweep town A away if the left bulldozer of B can move over to A pushing off all bulldozers of all towns on its way.

   Prove that there is exactly one town that cannot be swept away by any other one.

2. Let a, b, c be positive real numbers. Prove that

$$\sqrt[3]{abc}\left(\sum_{cyc}\frac{3a\sqrt{a}}{2a+b}\right) \leqslant \sum_{\text{cyc}}\frac{a^2(a+2b)}{\sqrt{b^2(b+c)+abc}}$$

3. Let $\triangle ABC$ be an acute triangle with orthocenter H. Points Y and Z are chosen on AC, AB such that $\angle HYC = \angle HZB = 60^\circ$. Let U be the circumcenter of $\triangle HYZ$, and let N be the nine-point center of $\triangle ABC$. Prove that A, U, N are collinear.[1]

[1] This problem was also given in TST 2011.

Thailand Team Selection Test for IMO 2016
IPST, Bangkok
29 April 2016

Day 11
Time: 4.5 hours

1. Let $ABCD$ be a convex quadrilateral such that $AC \perp BD$. Prove that there exist points P, Q, R, S, on the sides AB, BC, CD, DA respectively, such that $PR \perp QS$ and the area of the quadrilateral $PQRS$ is exactly half of that of the quadrilateral $ABCD$.

2. a) Prove that there does not exist a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $\gcd(f(m) + n, f(n) + m) = 1$ for all $m \neq n$

   b) Prove that there exists a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $\gcd(f(m)+n, f(n)+m) \leqslant 2$ for all $m \neq n$.

3. In a group of n students, each student has exactly k friends, where friendship is mutual. Any pair of students has exactly λ mutual friends. Suppose that the remainders of n, k, and λ when divided by 4 are 1, 2, and 2, respectively. Show that there exist four students A, B, C, D such that A, B and C are all friends with D but between A, B and C no two are friends.