Thailand Team Selection Test for IMO 2015
IPST, Bangkok
25 December 2014

Day 1
Time: 4.5 hours

1. Let $\square ABCD$ be a quadrilateral inscribed in a circle ω. The symmedians through B of $\triangle ABD$ and $\triangle CBD$ intersect ω again at points P and Q, respectively. Lines CP and AB intersect at X, and lines AQ and BC intersect at Y. Prove that the points X, D, Y are collinear.

   Note: the symmedian is the reflection of the median over the internal angle bisector.

2. A group of seven people has the property that any six of them can be seated in a round table so that any two neighbors are friends. Prove that all seven people can also be seated in a round table so that any two neighbors are friends. (Assume that friendship is mutual.)

3. Find all prime numbers $p \leqslant q \leqslant r$ for which there exist positive integers a, b, c, x, y such that
$$p^a q^b r^c = 7^{2x} - 2^{2y}7^{2x} - 10^{2x} + 2^{2x+2y}25^x.$$

4. Let a, b, c be positive integers such that $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Prove that
$$\frac{2a^2}{\sqrt{a^2+b^2}} + \frac{2b^2}{\sqrt{b^2+c^2}} + \frac{2c^2}{\sqrt{c^2+a^2}} \geqslant \sqrt{\frac{a^2+3}{2}} + \sqrt{\frac{b^2+3}{2}} + \sqrt{\frac{c^2+3}{2}}.$$

Thailand Team Selection Test for IMO 2015
IPST, Bangkok
26 December 2014

Day 2
Time: 4.5 hours

1. Find the least positive integer k for which there exist polynomials f, g with integer coefficients such that
$$f(X)(X+1)^{2014} + g(X)(X^{2014}+1) = k.$$

2. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f\big(x + f(x+2557) + 2557\big) = f(x+2557)$$
for all real numbers x.

3. Susan has three balls, initially with the numbers 56, 2014, and 2557 written on them, respectively. Each minute, Susan chooses one of the balls. If the chosen ball has number c, and the other two balls have numbers a and b, then Susan replaces the number c with the number $\frac{1}{c(a+b)}$.

   Is it possible for Susan to reach a point where the three balls have the numbers $56, 2015$, and 2558?

4. Let P be a point in the interior of a triangle ABC. The three cevians AA', BB', CC' of P divide the triangle into six triangles. Prove that the circumcenters of the six triangles are concyclic if and only if P is the centroid of $\triangle ABC$.

Thailand Team Selection Test for IMO 2015
IPST, Bangkok
25 January 2015

Day 3
Time: 4.5 hours

1. Let $P(x)$ be a polynomial with integer coefficients and let $x_1, x_2, \dots, x_{2015}$ be distinct integers such that
$$1 \leqslant P(x_1), P(x_2), \dots, P(x_{2015}) \leqslant 2014.$$
Prove that $P(x_1) = P(x_2) = \cdots = P(x_{2015})$.

2. What is the maximum number of squares in an 8×8 board that can be colored so that for each square in the board, at most one square adjacent to it is colored.

   Note: two squares are adjacent if they share a side.

3. Let $\triangle ABC$ be a triangle with $\angle ACB = 60^\circ$. Let D and E be points on sides BC and AC, respectively, so that $\frac{AE}{BD} = \frac{BC}{CE} - 1$. Let P be the intersection of AD and BE, and let $Q \neq P$ be the intersection of the circumcircles of $\triangle AEP$ and $\triangle BDP$. Prove that $QE \parallel BC$.

4. A non-empty finite set S of positive integers is given. For each nonempty subset $T = \{t_1, t_2, \dots, t_k\}$ of S, we define the *score* of T to be $(-2)^k \gcd(t_1, t_2, \dots, t_k)$. Prove that the sum of scores of all nonempty subsets of S is negative.

5. Let $n \geqslant 2$ be a positive integer. Let $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$ be distinct sets of positive integers. Suppose that the sets $S_A = \{a_i + a_j \mid 1 \leqslant i < j \leqslant n\}$ and $S_B = \{b_i + b_j \mid 1 \leqslant i < j \leqslant n\}$ are equal. What are the possible values of n?

Thailand Team Selection Test for IMO 2015
IPST, Bangkok
18 March 2015

Day 4
Time: 4.5 hours

1. Let $a, b_1, c_1, b_2, c_2, \ldots, b_n, c_n$ be real numbers such that the equation

   $$x^{2n} + ax^{2n-1} + ax^{2n-2} + \ldots + ax + 1 = (x^2 + b_1x + c_1)(x^2 + b_2x + c_2)\cdots(x^2 + b_nx + c_n)$$

   holds for all real numbers x. Prove that $c_1 = c_2 = \ldots = c_n = 1$.

2. Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$.

3. Let n be a positive integer. Each element of the set $\{1, 2, \ldots, n\}$ is colored with one of three colors so that each color is used at most $n/2$ times.

   Let A be the set of all quadruples (a, b, c, d) of elements of S such that $a + b + c + d$ is divisible by n ant a, b, c, d are all the same color.

   Let B be the set of all quadruples (a, b, c, d) of elements of S such that $a + b + c + d$ is divisible by n, a and b have the same color, and c and d have the same color, but different from that of a and b.

   Prove that $|B| \geqslant |A|$.

Thailand Team Selection Test for IMO 2015
IPST, Bangkok
19 March 2015

Day 5
Time: 4.5 hours

1. Determine all pairs (x, y) of positive integers such that
$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1.$$

2. Let $a, b, c \in [0, 1]$. Prove that
$$\frac{a+b}{\sqrt{2c^2+3}} + \frac{b+c}{\sqrt{2a^2+3}} + \frac{c+a}{\sqrt{2b^2+3}} \leqslant \frac{6}{\sqrt{5}}.$$

3. In $\triangle ABC$, H is the orthocenter, I is the incenter, and O is the circumcenter. Let K be the point where the incircle touches BC. If $IO \parallel BC$, then prove that $AO \parallel HK$.

Thailand Team Selection Test for IMO 2015
IPST, Bangkok
30 March 2015

Day 6
Time: 4.5 hours

1. A triangle $\triangle ABC$ with $AB > BC$ is inscribed into a circle ω. Points M and N are on the sides AB and BC, respectively, such that $AM = CN$. Let lines MN and AC intersect at K, and let S be the midpoint of arc AC containing B. The circle with diameter KS intersects line MN at T and intersects the angle bisector of $\angle MKA$ at D.

   a) Prove that $MT = TN$.

   b) Let P be the incenter of $\triangle CKN$ and let Q be the K-excenter of $\triangle AKM$. Prove that $DP = DQ$.

2. A group of students is separated into three classrooms. It is known that for each pair of students from different classrooms, among the students of the third classrooms there are exactly 10 students who are friends with both and exactly 10 students who are friends with neither. What is the total number of students in all three classrooms?

3. Consider all polynomials $P(x)$ with real coefficients that have the following property: for any two real numbers x and y one has

$$|y^2 - P(x)| \leqslant 2|x| \quad \text{if and only if} \quad |x^2 - P(y)| \leqslant 2|y|.$$

   Determine all possible values of $P(0)$.

Thailand Team Selection Test for IMO 2015
IPST, Bangkok
31 March 2015

Day 7
Time: 4.5 hours

1. Find all non-decreasing functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(1) = 1$ and

$$f\big(n + f(n) + 1\big) = f(n) + 1 \quad \text{and}$$
$$f(m)f(n) \leqslant f(2mn + m + n)$$

   for all positive integers m and n.

2. Find the necessary and sufficient conditions for a prime p to be of the form $p = x^2+xy+y^2$ for some integers x, y.

3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

   Prove that line BD is tangent to the circumcircle of triangle TSH.[1]

[1] Yes, this is IMO 2014/3.

Thailand Team Selection Test for IMO 2015
IPST, Bangkok
16 April 2015

Day 8
Time: 4.5 hours

1. Let $S = \{1, 2, \dots, 2015\}$. Each number in S is colored either red or blue. In an operation, we are allowed to choose a number in S, and change its color along with the colors of all numbers that are *not* relatively prime to it. Suppose that initially all of the numbers are red. Determine whether we can perform finitely many operations so that all of the numbers become blue.

2. Let n be an integer. Prove that there exists exactly one sequence of integers $x_0, x_1, x_2, \dots$ such that
$$n = x_0(-2015)^0 + x_1(-2015)^1 + x_2(-2015)^2 + \cdots$$
and $0 \leqslant x_i < 2015$ for each i.

3. Let $\square ABCD$ be a convex quadrilateral, and let M be a point inside the quadrilateral. Suppose that the projections of M onto the sides AB, BC, CD, DA lie on a circle with center O. Let N be the reflection of M over O. Prove that the projections of B onto the lines AM, AN, CM, CN also lie on a circle.

Thailand Team Selection Test for IMO 2015
IPST, Bangkok
17 April 2015

Day 9
Time: 4.5 hours

1. Let $\square ABCD$ be a convex quadrilateral such that $\angle DAB = 180^\circ - 2\angle BCD$. The incircle of $\triangle ABD$ is tangent to the sides AB and AD at P and Q, respectively. Prove that the circumcircle of $\triangle APQ$ is tangent to the circumcircle of $\triangle BCD$.

2. Determine all positive real numbers x, y, z, w that satisfy the following equation:

$$\frac{x^2 - yw}{y + 2z + w} + \frac{y^2 - zx}{z + 2w + x} + \frac{z^2 - wy}{w + 2x + y} + \frac{w^2 - xz}{x + 2y + z} = 0.$$

3. Determine all integers a, b such that $a^3 - a + 9 = 5b^2$.

Thailand Team Selection Test for IMO 2015
IPST, Bangkok
29 April 2015

Day 10
Time: 4.5 hours

1. Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of the triangle ABC.

2. For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1\leq i\leq n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i-th step he chooses x_i among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \leqslant cD$.

3. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(m) + n\big) + f(m) = f(n) + f(3m) + 2014$$

for all integers m and n.

Thailand Team Selection Test for IMO 2015
IPST, Bangkok
30 April 2015

Day 11
Time: 4.5 hours

1. Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

2. Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geq 1}$, defined by
$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor$$
are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

3. Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF . Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T , respectively. We call the pair (E, F) *interesting* if the quadrilateral $KSAT$ is cyclic.

   Suppose that the pairs (E_1, F_1) and (E_2, F_2) are interesting. Prove that $\dfrac{E_1E_2}{AB} = \dfrac{F_1F_2}{AC}$.