Thailand Team Selection Test for IMO 2014
IPST, Bangkok
26 December 2013

Day 1
Time: 4 hours

1. Let a, b, c be positive real numbers such that $a + b + c = abc$. Prove that

$$\frac{a(a+b)+b(b+c)+c(c+a)-3}{5} \geqslant \frac{ab^2+bc^2+ca^2}{a+b+c} \geqslant \sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}}.$$

2. Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function satisfying the following conditions:

   (i) $f(x + y) \leqslant f(x) + f(y)$ for all $x, y \in (0, \infty)$, and

   (ii) $f(2556x) = 2556f(x)$ for all $x \in (0, \infty)$.

   Prove that $f(x) = f(1) \cdot x$ for all $x \in (0, \infty)$.

3. In $\triangle ABC$, the incircle γ is tangent to the sides BC, CA and AB at A_1, B_1 and C_1, respectively. Let ω_A be the unique circle through B and C which is tangent to γ; the line through A_1 and the tangency point of γ and ω_A meets ω_A again at M_A. Points M_B and M_C are defined analogously. Prove that

   a) A_1M_A, B_1M_B, C_1MC concur at a single point, which we call K, and

   b) K, I, O are collinear and the distances between them satisfy

   $$\frac{KO}{KI} = \frac{R(M_AM_BM_C)}{r(ABC)},$$

   where $R(XYZ)$ and $r(XYZ)$, respectively, denote the circumradius and inradius of $\triangle XYZ$.

4. Let n be a positive integer. $2^{2n-1} + 1$ odd numbers are chosen between 2^{2n} and 2^{3n}. Prove that it is possible to choose two of them, say, x and y, so that x does not divide y^2 and y does not divide x^2.

Thailand Team Selection Test for IMO 2014
IPST, Bangkok
27 December 2013

Day 2
Time: 3 hours

1. Let $\triangle ABC$ be a scalene triangle with incenter I, and let I_b and I_c be its excenters opposite vertices B and C, respectively. Let D be the intersection point of the perpendiculars from I_b to AC and from I_c to AB. The angle bisectors of $\angle BI_bD$ and $\angle CI_cD$ are drawn intersecting at G, and the line through G parallel to AI intersects I_bI_c at H. Prove that the circle centered at G with radius GH is tangent to the circumcircle of $\triangle ABC$.

2. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be two sequences of integers such that

$$|a_{n+2} - a_n| \leqslant 2 \quad \text{and} \quad a_m + a_n = b_{m^2+n^2}$$

for all positive integers m, n. Prove that the sequence $a_1, a_2, \ldots$ contains at most six distinct values.

3. Find all odd primes p such that both

$$1 + p + p^2 + p^3 + \cdots + p^{p-2} + p^{p-1} \quad \text{and} \quad 1 - p + p^2 - p^3 + \cdots - p^{p-2} + p^{p-1}$$

are primes.

Thailand Team Selection Test for IMO 2014
IPST, Bangkok
25 January 2014

Day 3
Time: 4 hours

1. Let M be an arbitrary point on the circumcircle of $\triangle ABC$. The tangents from this point to the incircle of $\triangle ABC$ meets BC at X_1 and X_2. Prove that the second intersection of the circumcircles of $\triangle MX_1X_2$ and $\triangle ABC$ (distinct from M) coincides with the tangency point of the A-mixtilinear incircle and the circumcircle of $\triangle ABC$.

   Note: as usual, the A-mixtilinear incircle of $\triangle ABC$ is the circle tangent to lines AB, AC and the circumcircle of $\triangle ABC$ internally.

2. Let a, b, c be positive real numbers. Prove that

$$\sqrt[3]{a^2b^2}+\sqrt[3]{b^2c^2}+\sqrt[3]{c^2a^2} \leqslant \frac{a(b+c)}{\sqrt[3]{a^4}+\sqrt[3]{b^2c^2}}+\frac{b(c+a)}{\sqrt[3]{b^4}+\sqrt[3]{c^2a^2}}+\frac{c(a+b)}{\sqrt[3]{c^4}+\sqrt[3]{a^2b^2}} \leqslant \sqrt[3]{a^4}+\sqrt[3]{b^4}+\sqrt[3]{c^4}.$$

3. In a 17×17 matrix, each entry is an integer from $1, 2, \ldots, 17$, and each integer appears in exactly 17 entries. Prove that it is possible to find a row or a column of the matrix which contains at least 5 different numbers.

4. Let a, b, c be positive integers for which $ac = b^2 + b + 1$. Prove that the equation

$$ax^2 - (2b+1)xy + cy^2 = 1$$

   has a solution in the set of positive integers.

Thailand Team Selection Test for IMO 2014
IPST, Bangkok
26 January 2014

Day 4
Time: 4 hours

1. Let $\triangle ACB$ be an acute triangle with circumcenter O, orthocenter H, and nine-point center N. Let P be the second intersection of AO and the circumcircle of $\triangle BOC$, and let Q be the reflection of A over BC. Show that the midpoint of segment PQ lies on line AN.

2. Every two of n towns in a country are connected by a one way or two way road. It is known that for every k towns, there exists a round trip passing through each of these k towns exactly once. Find the maximal possible number of one way roads in this country.

3. Find all positive integers n for which
$$\left(1^4 + \frac{1}{4}\right)\left(2^4 + \frac{1}{4}\right)\cdots\left(n^4 + \frac{1}{4}\right)$$
is the square of a rational number.

4. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying the equation
$$f(xy) + f(x + y) = 1 + f(x)f(y)$$
for all rational numbers x, y.

Thailand Team Selection Test for IMO 2014
IPST, Bangkok
17 March 2014

Day 5
Time: 4.5 hours

1. Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

2. Let a, b, c, d be positive reals satisfying $a^3 + b^3 + c^3 + d^3 \leqslant 4$. Prove that

$$\frac{1}{\sqrt{abc}} + \frac{1}{\sqrt{abd}} + \frac{1}{\sqrt{acd}} + \frac{1}{\sqrt{bcd}} \geqslant a + b + c + d.$$

3. Let $R(x) = \frac{F(x}{G(x)}$ be a rational function where $F(x)$ and $G(x)$ are polynomials with integer coefficients, chosen so that $F(x)$ and $G(x)$ has no common root modulo p for all primes p. Let n be a positive integer, and define the rational function

$$Q(x) = \underbrace{R(R(\cdots(R}_{n}(x)))).$$

Suppose that there is an integer k such that $Q(k) = k$. Prove that $R(R(k)) = k$.

2 points (out of 7) are given for proving the case $G(x) \equiv 1$.

Thailand Team Selection Test for IMO 2014
IPST, Bangkok
18 March 2014

Day 6
Time: 4.5 hours

1. A function $f : \mathbb{Q} \to \mathbb{Z}$ is given. Prove that there exist two distinct rational numbers p and q such that
$$\frac{f(p) + f(q)}{2} \leqslant f\left(\frac{p+q}{2}\right).$$

2. Let $\square ABCD$ be a quadrilateral such that AC bisects $\angle BAD$ and BD bisects angle $\angle ABC$. A rhombus $\square KLMN$ is inscribed in $\square ABCD$ so that all vertices of the rhombus lie on different sides of $\square ABCD$. Prove that the non-obtuse angle φ of the rhombus satisfy $\varphi \leqslant \max\{\angle BAD, \angle ABC\}$.

3. On a table there are n piles of books, each having at least one book. Peter comes along and rearranges the books into $n+1$ piles, each having at least one book. Call a book *lucky* if it ends up in a pile with fewer books than before. Prove that there are at least two lucky books.

Thailand Team Selection Test for IMO 2014
IPST, Bangkok
25 March 2014

Day 7
Time: 4.5 hours

1. Find all functions $f : \mathbb{Z} \to \mathbb{R}$ such that

$$f(k) \leqslant 2557 \quad \text{and } f(k) \leqslant \frac{f(k-1)+f(k+1)}{2}$$

   for all integers k.

2. Determine whether there exists an infinite sequence of nonzero digits $a_1, a_2, a_3, \cdots$ and a positive integer N such that for every integer $k > N$, the number $\overline{a_k a_{k-1} \cdots a_1}$ is a perfect square.

3. Let $\triangle ABC$ with a scalene triangle whose incircle $\odot(I, r)$ (i.e. centered at I with radius r) touches the sides BC, CA, AB at X, Y, Z respectively. Let X_1, Y_1, Z_1, respectively, be the images of X, Y, Z under the homothety $h(I, 2r)$. (In other words, X_1, Y_1, Z_1 lie on the rays $\overrightarrow{IX}, \overrightarrow{IY}, \overrightarrow{IZ}$, respectively, such that $IX_1 = IY_1 = IZ_1 = 2r$.) Prove that

   a) the lines AX_1, BY_1, CZ_1 pass through a single point, which we call Q, and

   b) if O is the circumcircle of $\triangle ABC$, and P is the intersection point line OI and the reflection of line AQ over line AI, then $\angle PCI = \angle ICQ$.

Thailand Team Selection Test for IMO 2014
IPST, Bangkok
26 March 2014

Day 8
Time: 4.5 hours

1. Let a, b and c be positive integers such that

$$0 < a^2 + b^2 - abc \leqslant c.$$

   Prove that $a^2 + b^2 - abc$ is a perfect square.

2. Prove that the nine-point circle of the triangle formed by the diagonals of a complete quadrilateral passes through the Miquel point of that quadrilateral.

   Note: if a complete quadrilateral is formed by points A, B, C, D, $E = AB \cap CD$, and $F = AD \cap BC$ then its three diagonals are AC, BD, and EF, and its Miquel point is the unique point lying on the circumcircles of $\triangle ADE, \triangle BCE, \triangle ABF$, and $\triangle CDF$.

   Hint, as given in the test: consider the focus of a parabola tangent to the quadrilateral.

3. 3.1 *(4 points)* Prove that every convex polyhedron without a quadrilateral or pentagonal face must have at least 4 triangular faces.

   3.2 *(3 points)* Let P be a set consisting of 2557 distinct prime numbers. Let A be the set of all possible products of 1278 elements of P, and B be the set of all possible products of 1279 elements of P. Prove that there exists a one-to-one function f from A to B with the property that a divides $f(a)$ for all $a \in A$.

Thailand Team Selection Test for IMO 2014
IPST, Bangkok
30 March 2014

Day 9
Time: 4.5 hours

1. Determine all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$x^2 + f(y) \mid xf(x) + y$$

for all positive integers x and y.

2. Let G be a finite undirected graph. The following two operations are allowed to be performed on G:

   (i) If a vertex V has odd degree, then we can delete V and all edges connected to it.

   (ii) We can create a copy V' of every vertex V. In this operation, the two copies V' and W' are connected by an edge if and only if the original vertices V and W are connected by an edge, and each copy V' has an edge connecting it to the original vertex V. No other edges appear or disappear.

   Prove that it is possible to apply a finite of operations on G so that the resulting graph contains no edges.

3. A tangential quadrilateral $\square ABCD$ which is not a trapezoid is given. The extensions of sides AD and BC intersect at E and the extensions of sides AB and CD intersect at F, so that exactly one of $\triangle AEF$ and $\triangle CEF$ is outside $\square ABCD$. Let the incircle of $\triangle AEF$ be tangent to lines AD and AB at K and L, respectively, and let the incircle of $\triangle CEF$ be tangent to lines BC and CD at M and N, respectively.

   a) Prove that K, L, M, N lie on a circle.

   b) Prove that A, B, C, D lie on a circle if and only if $KN \perp LM$.

   Note: a tangential quadrilateral *is a quadrilateral with an inscribed circle, i.e. a circle contained in the quadrilateral which is tangent to all four of its sides.*

Thailand Team Selection Test for IMO 2014
IPST, Bangkok
31 March 2014

Day 10
Time: 4.5 hours

1. Let x, y, z be positive real numbers. Prove that

$$\frac{3x}{4x^2+4y^2+z^2}+\frac{3y}{4y^2+4z^2+x^2}+\frac{3z}{4z^2+4x^2+y^2} \leqslant \sqrt{\frac{1}{x^2+xy+y^2}+\frac{1}{y^2+yz+z^2}+\frac{1}{z^2+zx+x^2}}.$$

2. A cyclic quadrilateral $\square ABCD$ is given. Let M be the set of the 16 centers of all incircles and excircles of $\triangle BCD, \triangle ACD, \triangle ABD$ and $\triangle ABC$. Prove that there exist two sets K and L, each consisting of four parallel lines, such that any line in $K \cup L$ contains exactly four points of M.

3. Find all integers x and y satisfying the equation

$$xy - 7\sqrt{x^2+y^2} = 1.$$

Thailand Team Selection Test for IMO 2014
IPST, Bangkok
21 April 2014

Day 11
Time: 4.5 hours

1. A scalene triangle $\triangle ABC$ is given with circumcircle ω and circumcenter O. Let M be the midpoint of BC, and assume $M \neq O$. The circumcircle of $\triangle OAM$ intersects ω again at $D \neq A$.

   a) The intersection point of the tangent lines to ω at A and D lies on line BC.

   b) The triangles $\triangle AMB, \triangle ACD$ and $\triangle DMB$ are similar.

2. For each positive integer k, let $L(k)$ denote the largest prime divisor of k. Prove that there exist infinitely many positive integers n such that

$$L(n^4 + n^2 + 1) = L\big((n+1)^4 + (n+1)^2 + 1\big).$$

3. Let $\mathbb{Z}_{\geqslant 0}$ be the set of all nonnegative integers. Find all the functions $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ satisfying the relation

$$f(f(f(n))) = f(n+1) + 1$$

   for all $n \in \mathbb{Z}_{\geqslant 0}$.

Thailand Team Selection Test for IMO 2014
IPST, Bangkok
22 April 2014

Day 12
Time: 4.5 hours

1. Prove that in any set of 2000 distinct real numbers there exist two pairs $a > b$ and $c > d$ with $a \neq c$ or $b \neq d$, such that
$$\left|\frac{a-b}{c-d} - 1\right| < \frac{1}{100000}.$$

2. Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

3. In some country several pairs of cities are connected by direct two-way flights. It is possible to go from any city to any other by a sequence of flights. The distance between two cities is defined to be the least possible numbers of flights required to go from one of them to the other. It is known that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

Thailand Team Selection Test for IMO 2014
IPST, Bangkok
29 April 2014

Day 13
Time: 4.5 hours

1. A certain graph has mn edges, and it is known that its edges can be painted using m colors in such a way that for any given vertex of the graph, all edges adjacent to the vertex have different colors. Prove that the edges can be painted using m colors so that for any given vertex of the graph, all edges adjacent to the vertex have different color, and, in addition, there are exactly n edges of each color.

2. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{5+a^4+b^3}+\frac{b}{5+b^4+c^3}+\frac{c}{5+c^4+a^3} \leqslant \frac{1}{7}\left(\sqrt{\frac{a^2+2b^2}{a^2+bc+ca}}+\sqrt{\frac{b^2+2c^2}{b^2+ca+ab}}+\sqrt{\frac{c^2+2a^2}{c^2+ab+bc}}\right).$$

3. A given convex polygon can cover any triangle whose side lengths are at most 1. Prove that the area of the convex polygon is at least $\frac{1}{2}\cos 10^\circ$.

Thailand Team Selection Test for IMO 2014
IPST, Bangkok
30 April 2014

Day 14
Time: 4.5 hours

1. Prove that every positive rational number r can be represented as
$$r = \frac{a^3 + b^3}{c^3 + d^3}$$
where a, b, c, d are positive integers.

2. Let I, I_A and O be the incenter, A-excenter and circumcenter of a triangle $\triangle ABC$, respectively. In the circumcircle ω of $\triangle ABC$, let M be the midpoint of the arc BC not containing A, and K the midpoint of an arc AM. Let $P \neq K$ be the second intersection point of KI and ω, and let $Q \neq K$ be the second intersection point of KI_A and ω. Suppose that the lines AM and BC intersect at N. Prove that P, Q, N are collinear.

3. Prove that every positive integer is the difference of two coprime composite positive integers.