Thailand Team Selection Test for IMO 2013
IPST, Bangkok
24 December 2012

Day 1
Time: 4.5 hours

1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\big(x + f(y)\big) - f(x) = \big(x + f(y)\big)^{2013} - x^{2013}$$

for all real numbers x and y.

2. Let n be a positive integer and let $G = \{z \in \mathbb{C} \mid z^n = 1\}$. Determine all functions $f : G \to \mathbb{Z}$ satisfying the following two conditions:

   (i) $f(z) = 1$ if and only if $z = 1$, and

   (ii) $f(z^k) = \frac{f(z)}{\gcd(f(z),k)}$ for all $z \in G$ and all positive integers n.

3. In $\triangle ABC$, the incircle centered at I touches sides BC, CA and AB at D, E and F, respectively. A circle k cuts segments EF, FD and DE at $\{X_1, X_2\}, \{X_3, X_4\}$ and $\{X_5, X_6\}$, respectively. Suppose that lines X_1X_4, X_2X_5 and X_3X_6 all pass through the center G of k.

   a) Prove that points A, D and G are collinear.

   b) Let the two lines through G parallel to DE and DF intersect line BC at P and Q. Prove that $IP = IQ$.

Thailand Team Selection Test for IMO 2013
IPST, Bangkok
17 January 2013

Day 2
Time: 4.5 hours

1. Determine all increasing functions $f : \mathbb{Z}^+ \to \mathbb{R}$ such that
$$f(mn) = f(m)f(n)$$
for all positive integers m and n.

2. Determine all ordered pairs (x, y) of positive integers satisfying the equation
$$x^3 + y^3 = 4(x^2y + y^2x - 5).$$

3. A stone is placed on the coordinate plane. If the stone is at position (x, y), then
   (i) for each positive integer z, the stone can be moved to $(x - z, y - z)$;
   (ii) the stone can also be moved to either $(3x, y)$ or $(3y, x)$.

   Find all pairs of positive integers (m, n) for which it is possible to move the stone from (m, n) to $(0, 0)$ in a finite number of moves.

Thailand Team Selection Test for IMO 2013
IPST, Bangkok
21 January 2013

Day 3
Time: 4.5 hours

1. Let S be a set of students with $|S| \geqslant 4$. Suppose that there exists a positive integer m with $3 \leqslant m \leqslant |S|-1$ such that for each $A \subseteq S$ with $|A| = m$, there is a unique student who is friends with every student in A. (A student cannot be friends with themself.) Prove that either[1]

   (i) there exists a subset $B \subseteq S$ with $|B| = m+1$ such that the students in B are pairwise friends, or

   (ii) $|S| = m+1$.

2. Let M be the midpoint of arc BC not containing A on the circumcircle of a given triangle $\triangle ABC$. Let I be the incenter of $\triangle ABC$, and let E and F be the projections of I onto MB and MC, respectively. Prove that $IE + IF \leqslant AM$.

3. Find all ordered pairs (a, b) of positive integers such that n divides $a^n + b^{n+1}$ for all positive integers n.[2]

[1] Unclear whether we are supposed to prove *either* (i) *or* (ii) or (i) *and* (ii).
[2] China Western MO 2011

Thailand Team Selection Test for IMO 2013
IPST, Bangkok
23 January 2013

Day 4
Time: 4.5 hours

1. Let x, y, z be positive reals. Show that

$$\frac{x^2}{y(x+y)+z(x+z)}+\frac{y^2}{z(y+z)+x(y+z)}+\frac{z^2}{x(z+x)+y(z+y)} \geqslant \frac{x}{(x+y)+(x+z)}+\frac{y}{(y+z)+(y+x)}+\frac{z}{(z+x)+(z+y)}.$$

2. Let O and I be the circumcenter and incenter of a scalene triangle $\triangle ABC$ respectively. The incircle of $\triangle ABC$ touches sides BC, CA and AB at D, E and F, respectively. Let AP, BQ and CR be the angle bisectors of $\triangle ABC$ so that P, Q and R lie on BC, CA and AB respectively. Let the reflections of line OI across lines DE and DF intersect at X. Prove that P, Q, R, X lie on a circle.

3. There are $k \geqslant 2$ piles of coins having $n_1, n_2, \ldots, n_k$ coins respectively. A move consists in choosing two piles having a and b coins respectively, where $a \geqslant b$, and transferring b coins from the first pile to the second one. Find the necessary and sufficient condition for $n_1, n_2, \ldots, n_k$, such that there exists a succession of moves through which all coins are transferred to the same pile.[3]

[3] Romania National Olympiad 2012

Thailand Team Selection Test for IMO 2013
IPST, Bangkok
24 January 2013

Day 5
Time: 4.5 hours

1. Let $P(x) \in \mathbb{Q}[x]$ be an irreducible polynomial (over $\mathbb{Q}[x]$.) Suppose that there exists an irrational number α such that $P(\alpha) = P(-\alpha) = 0$. Prove that there exists an irreducible polynomial $Q(x) \in \mathbb{Q}[x]$ such that $P(x) = Q(x^2)$.

2. Determine all positive integers n such that[4]

$$\left\lfloor \frac{1000000}{n} \right\rfloor - \left\lfloor \frac{1000000}{n+1} \right\rfloor = 1.$$

3. Let $\triangle ABC$ be a triangle with $AB > AC$. The incircle of $\triangle ABC$ touches its sides BC, CA and AB at D, E and F, respectively. The angle bisector of $\angle BAC$ cuts DE and DF at K and L, respectively. Let M be the midpoint of BC and let H be the foot of altitude from A to BC. Prove that $\angle MLK = \angle MHK$.

[4] Adapted from Japan MO 2012 Preliminary Round

Thailand Team Selection Test for IMO 2013
IPST, Bangkok
16 March 2013

Day 6
Time: 4.5 hours

1. Let $P_1, P_2, \ldots, P_n$ ($n \geqslant 3$) be points on a unit circle. Suppose that for any point Q on the unit circle, the product of the distances from Q to $P_1, P_2, \ldots, P_n$ is less than or equal to 2. Prove that $P_1, P_2, \ldots, P_n$ are vertices of a regular n-gon.

2. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the functional equation
$$f\left(2x + \frac{1}{1+x+y}\right) = f(x) + f\left(x + \frac{1}{1+x+y}\right)$$
for all positive reals x and y.

3. Let S be the set of all positive integers with exactly 11 digits (when written in base 10), and let A be a subset of S. An element $x \in A$ is called *lonely* if for all $y, z \in A$, $y + z$ does not divide x. Suppose that A has at most 10 lonely numbers. What is the maximum possible number of elements in A?

Thailand Team Selection Test for IMO 2013
IPST, Bangkok
20 March 2013

Day 7
Time: 4.5 hours

1. Find all triples (x, y, z) of positive integers such that $x \leqslant y \leqslant z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

2. Let a, b, c be positive reals satisfying $abc = 1$. Prove that

$$\frac{1}{1 + a^4 + (b^2 + 1)^2} + \frac{1}{1 + b^4 + (c^2 + 1)^2} + \frac{1}{1 + c^4 + (a^2 + 1)^2} \leqslant \frac{a}{2b + c + 3} + \frac{b}{2c + a + 3} + \frac{c}{2a + b + 3}.$$

3. Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

Thailand Team Selection Test for IMO 2013
IPST, Bangkok
31 March 2013

Day 8
Time: 4.5 hours

1. In a 2556×2556 square table some cells are white and the remaining ones are red. Let T be the number of triples (C_1, C_2, C_3) of cells, the first two in the same row and the last two in the same column, with C_1, C_3 white and C_2 red. Find the maximum value T can attain.

2. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\big(f(x+y) + f(x-y)\big)^2 = 4f(x)^2$$

for all real numbers x, y.

3. Let x and y be positive integers. If $x^{2^n} - 1$ is divisible by $2^n y + 1$ for every positive integer n, prove that $x = 1$.

Thailand Team Selection Test for IMO 2013
IPST, Bangkok
2 April 2013

Day 9
Time: 4.5 hours

1. On a plane, $n \geqslant 4$ parallel line segments lie so that any three of the n segments can be cut by a single line. Prove that all n segments can be cut by a single line.

2. Let $a, b, c > 0$. Prove that

$$32\left(\frac{1}{7+(a-3)^2}+\frac{1}{7+(b-3)^2}+\frac{1}{7+(c-3)^2}\right) \leqslant \frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}+6.$$

3. In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

Thailand Team Selection Test for IMO 2013
IPST, Bangkok
3 April 2013

Day 10
Time: 4.5 hours

1. Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

2. Let $(a_1, a_2, \dots, a_{2n})$ be a permutation of $\{1, 2, \dots, 2n\}$ such that the values of $|a_{i+1} - a_1|$ are distinct for all $i \in \{1, 2, \dots, 2n-1\}$. Prove that $a_1 - a_{2n} = n$ if and only if $1 \leqslant a_{2k} \leqslant n$ for each $k = 1, 2, \dots, n$.

3. Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \dots$ is unbounded.

Thailand Team Selection Test for IMO 2013
IPST, Bangkok
5 April 2013

Day 11
Time: 4.5 hours

1. Does there exist a partition of $\mathbb{Q}$ into three non-empty subsets A, B and C such that the sets $A+B, B+C$ and $C+A$ are disjoint?

   Note: as usual, $X+Y$ denotes the sumset $\{x+y \mid x \in X, y \in Y\}$.

2. An integer a is called *fossil* if the equation $(m^2+n)(n^2+m) = a(m-n)^3$ has a solution over the positive integers.

   a) Prove that there are at least 500 fossil integers in the set $\{1, 2, \ldots, 2012\}$.

   b) Decide whether $a = 2$ is fossil.

3. Let S be a finite set of positive integers whose smallest and largest elements are relatively prime. Let S_n be the set of all positive integers which can be expressed as the sum of at most n (not necessarily distinct) elements of S. Let a be the largest element of S. Prove that there exists a positive integer k such that for all positive integers $m > k$, $|S_{m+1}| - |S_m| = a$.