Thailand Team Selection Test for IMO 2012
IPST, Bangkok
25 January 2012

Day 1
Time: 4.5 hours

1. Let n be a positive integer $n \geqslant 2$. For a given permutation σ of $\{1, 2, \ldots, n\}$, write $\sigma = C_1 C_2 \ldots C_k$ as a product of k disjoint cycles of length $c_1, c_2, \ldots, c_k$, respectively. Then σ is called an *even permutation* or an *odd permutation* of degree n according to the parity of $\sum_{i=1}^{k}(c_i - 1)$. Prove that

   a) there are exactly $n!/2$ even permutations of degree n, and

   b) if $n \geqslant 3$ then a permutation σ of $\{1, 2, \ldots, n\}$ is an even permutation of degree n if and only if there exists 3-cycles $C_1, C_2, \ldots, C_\ell$ such that $\sigma = C_1 C_2 \ldots C_\ell$.

2. Solve the equation
$$\lfloor x^2 \rfloor - \lfloor -x^2 \rfloor - 8\lfloor x \rfloor + 2 = 0$$
over the set of real numbers.

3. Let M be the midpoint of side BC of a triangle $\triangle ABC$. Let S and T be points on BM and CM, respectively, such that $SM = MT$. Let P and Q be points on AT and AS, respectively, such that $\angle PST = \angle BAS$ and $\angle QTS = \angle CAT$. Suppose that lines BQ and PC intersect at a point R. Show that $RM \perp BC$.

Thailand Team Selection Test for IMO 2012
IPST, Bangkok
26 January 2012

Day 2
Time: 4.5 hours

1. Does there exist a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f(n+7) = f(f(n)) + f(n+5)$$

for all positive integers n?

2. Let $\triangle ABC$ be a non-equilateral triangle, and let I and O be its incenter and circumcenter, respectively. Prove that $\angle AIO \leqslant 90^\circ$ if and only if $2BC \leqslant AB + AC$, and that both inequalities become equalities simultaneously

3. A light bulb is placed on each vertex of a regular n-gon whose sides and diagonals are all drawn. In the beginning, k light bulbs are turned on. In each step, it is allowed to erase a side or a diagonal and change the state of one of the two bulbs at the endpoints of the side/diagonal. Determine all possible values of k for which it is always possible to reach the state where all light bulbs are on, and all diagonals are erased.

Thailand Team Selection Test for IMO 2012
IPST, Bangkok
4 February 2012

Day 3
Time: 4.5 hours

1. Let $G(V, E)$ be a simple graph with n vertices and m edges. For each vertex $x \in V$, the *degree* of x, denoted by $d(x)$, is the number of edges in E having x as an endpoint. Suppose that G does not contain the 4-cycle C_4 as a subgraph. Prove that

   a) $\displaystyle\sum_{x\in V}\binom{d(x)}{2} \leqslant \binom{n}{2}$, and

   b) $m \leqslant \dfrac{n\left(1+\sqrt{4n-3}\right)}{4}$.

2. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients with $P(x)$ being monic. Prove that there exists a polynomial $R(x)$ with integer coefficients such that $P(x)$ divides $R(Q(x))$.

3. Find, with proof, the least odd integer $a > 5$ for which there exists positive integers m_1, m_2, n_1, n_2 such that

$$a = m_1^2 + n_1^2, \quad a^2 = m_2^2 + n_2^2, \quad \text{and} \quad m_1 - n_1 = m_2 - n_2 \text{ is divisible by 3.}$$

Thailand Team Selection Test for IMO 2012
IPST, Bangkok
5 February 2012

Day 4
Time: 4.5 hours

1. Let a, b, c be non-negative real numbers such that $a+b+c \leqslant 3$. Prove that

$$\frac{1}{4}a^2b(b\sqrt{c}+3)^2+\frac{1}{4}b^2c(c\sqrt{a}+3)^2+\frac{1}{4}c^2a(a\sqrt{b}+3)^2 \leqslant abc(ab^2+bc^2+ca^2)+3(a^2b+b^2c+c^2a) \leqslant 12$$

2. a) Find a continuous, strictly decreasing function $f:\mathbb{R}\to\mathbb{R}$ such that $f(f(x)) = 2012x+2555$ for all real numbers x.

   b) Find all continuous, strictly decreasing functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(f(x)) = x+2555$ for all real numbers x.

3. Let $\triangle ABC$ be an acute-angled triangle with circumcircle $\odot O$ centered at O. Let P be a point inside $\triangle ABC$ and let lines AP, BP, CP intersect $\odot O$ again at A', B', C', respectively. Let B_c, C_b, A_b, B_a, C_a and A_c, respectively, denote the circumcenters of $\triangle PBC', \triangle PCB', \triangle PAB', \triangle PBA', \triangle PCA'$ and $\triangle PAC'$. Prove that lines B_cC_b, A_bB_a and C_aA_c concur at the midpoint of segment OP.

Thailand Team Selection Test for IMO 2012
IPST, Bangkok
28 March 2012

Day 5
Time: 4.5 hours

1. Does there exist a convex equilateral n-gon whose vertices all lie on the curve $y = x^2$, where
   a) $n = 2012$;
   b) $n = 2013$?

2. Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy
$$g(f(x+y)) = f(x) + (2x+y)g(y)$$
for all real numbers x and y.

3. Let p be an odd prime number. For every integer a, define the number
$$S_a = \frac{a}{1} + \frac{a^2}{2} + \cdots + \frac{a^{p-1}}{p-1}.$$
Let m and n be integers such that
$$S_3 + S_4 - 3S_2 = \frac{m}{n}.$$
Prove that p divides m.

Thailand Team Selection Test for IMO 2012
IPST, Bangkok
31 March 2012

Day 6
Time: 4.5 hours

1. Let a, b and c be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2 + b^2 + c^2 = 3$. Prove that
$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geqslant \frac{3}{(abc)^2}.$$

2. Let $n > 2$ be a positive integer and let $\Phi_n(x)$ be the n^{th} cyclotomic polynomial.
   a) Prove that $\Phi_n(a) > 0$ for all real numbers a.
   b) Find, with proof, the values of $\Phi_n(1)$ and $\Phi_n(-1)$.

3. Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

Thailand Team Selection Test for IMO 2012
IPST, Bangkok
4 April 2012

Day 7
Time: 4.5 hours

1. For any integer $d > 0$, let $f(d)$ be the smallest possible integer that has exactly d positive divisors (so for example we have $f(1) = 1, f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f\left(2^k\right)$ divides $f\left(2^{k+1}\right)$.

2. Let G be a graph with vertex set V and clique number $\omega(G) = 100$. Prove that there is a partition of V into sets V_1 and V_2 such that $\omega(G[V_1]) = \omega(G[V_2]) = 50$, where $G[V_1]$ and $G[V_2]$ are the subgraphs of G induced by V_1 and V_2 respectively.[1]

   Note: the clique number of a graph is the size of its largest clique.

3. Let ABC be a triangle with incentre I and circumcircle ω. Let D and E be the second intersection points of ω with AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and B meet at a point K. Prove that the three lines AE, BD and KP are either parallel or concurrent.

[1] IMO 2007/3

Thailand Team Selection Test for IMO 2012
IPST, Bangkok
5 April 2012

Day 8
Time: 4.5 hours

1. Prove that there exists 1000 consecutive positive integers containing exactly 25 primes.

2. Let $\Im$ be a convex figure. Prove that there is a point C in $\Im$ such that for any chord AB of $\Im$ through C, we have
$$|AC| \leqslant \frac{2}{3}|AB|.$$
Note: a chord *of a convex figure is a line segment with endpoints on the boundary of the figure.*

3. Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy
$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$
for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\dots f}_{k}(n)\dots))$.

Thailand Team Selection Test for IMO 2012
IPST, Bangkok
19 April 2012

Day 9
Time: 3 hours

1. A positive integer n is given. Prove that there are infinitely many monic polynomials of degree n with integer coefficients whose zeros all lie in the unit disc $\{z \in \mathbb{C} : |z| \leqslant 1\}$. Furthermore, show that all zeros of such polynomials have modulus zero or one.

2. Let a and b be positive integers. Suppose that the cyclotomic polynomials Φ_a and Φ_b satisfy

   $$\gcd\left(\Phi_a(t), \Phi_b(t)\right) > 1$$

   for some positive integer k. Prove that $\frac{a}{b}$ is a prime power, that is, $\frac{a}{b} = p^k$ for a prime number p and an integer k.

Thailand Team Selection Test for IMO 2012
IPST, Bangkok
26 April 2012

Day 10
Time: 4.5 hours

1. Let a, b, c be positive real numbers. Prove that

$$\frac{a\sqrt{b^3+c^3}}{b^2+c^2}+\frac{b\sqrt{c^3+a^3}}{c^2+a^2}+\frac{c\sqrt{a^3+b^3}}{a^2+b^2} \geqslant \frac{1}{\sqrt{2\left(\frac{a^5+b^5}{ab(a+b)}+\frac{b^5+c^5}{bc(b+c)}+\frac{c^5+a^5}{ca(c+a)}\right)}}.$$

2. For each positive integer k, let $t(k)$ be the largest odd divisor of k. Determine all positive integers a for which there exists a positive integer n, such that all the differences

$$t(n+a)-t(n); t(n+a+1)-t(n+1), \ldots, t(n+2a-1)-t(n+a-1)$$

are divisible by 4.

3. Let $\Im$ be a collection of subsets of $\{1, 2, \ldots, n\}$ such that

   (i) every element of $\Im$ is a 3-element set, and

   (ii) the intersection of any two elements of $\Im$ contains at most one element.

   Let f denote the maximum number of elements that $\Im$ can have. Prove that

$$n^2-4n \leqslant 6f \leqslant n^2-n.$$

Thailand Team Selection Test for IMO 2012
IPST, Bangkok
27 April 2012

Day 11
Time: 4.5 hours

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y)f(x-y) = (x-y)g(x+y)$$

for all real numbers x and y.

2. Let ABC be a triangle with $AB = AC$ and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B and C at the point E inside the triangle ABC. The line BD intersects the circle through A, E and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incentre of triangle KAB.

3. On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of k?