Thailand Team Selection Test for IMO 2012 IPST, Bangkok 25 January 2012

Day 1 Time: 4.5 hours

- 1. Let *n* be a positive integer $n \ge 2$. For a given permutation σ of $\{1, 2, ..., n\}$, write $\sigma = C_1 C_2 ... C_k$ as a product of *k* disjoint cycles of length $c_1, c_2, ..., c_k$, respectively. Then σ is called an *even permutation* or an *odd permutation* of degree *n* according to the parity of $\sum_{i=1}^{k} (c_i 1)$. Prove that
 - a) there are exactly n!/2 even permutations of degree n, and
 - b) if $n \ge 3$ then a permutation σ of $\{1, 2, ..., n\}$ is an even permutation of degree n if and only if there exists 3-cycles $C_1, C_2, ..., C_\ell$ such that $\sigma = C_1 C_2 ... C_\ell$.
- 2. Solve the equation

$$\lfloor x^2 \rfloor - \lfloor -x^2 \rfloor - 8 \lfloor x \rfloor + 2 = 0$$

over the set of real numbers.

3. Let M be the midpoint of side BC of a triangle $\triangle ABC$. Let S and T be points on BM and CM, respectively, such that SM = MT. Let P and Q be points on AT and AS, respectively, such that $\angle PST = \angle BAS$ and $\angle QTS = \angle CAT$. Suppose that lines BQ and PC intersect at a point R. Show that $RM \perp BC$.



Day 2 Time: 4.5 hours

1. Does there exist a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f(n+7) = f(f(n)) + f(n+5)$$

for all positive integers n?

- 2. Let $\triangle ABC$ be a non-equilateral triangle, and let I and O be its incenter and circumcenter, respectively. Prove that $\angle AIO \leq 90^{\circ}$ if and only if $2BC \leq AB + AC$, and that both inequalities become equalities simultaneously
- 3. A light bulb is placed on each vertex of a regular *n*-gon whose sides and diagonals are all drawn. In the beginning, *k* light bulbs are turned on. In each step, it is allowed to erase a side or a diagonal and change the state of one of the two bulbs at the endpoints of the side/diagonal. Determine all possible values of *k* for which it is always possible to reach the state where all light bulbs are on, and all diagonals are erased.



Thailand Team Selection Test for IMO 2012 IPST, Bangkok 4 February 2012

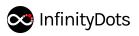
Day 3 Time: 4.5 hours

1. Let G(V, E) be a simple graph with n vertices and m edges. For each vertex $x \in V$, the *degree* of x, denoted by d(x), is the number of edges in E having x as an endpoint. Suppose that G does not contain the 4-cycle C_4 as a subgraph. Prove that

a)
$$\sum_{x \in V} {d(x) \choose 2} \leq {n \choose 2}$$
, and
b) $m \leq \frac{n(1 + \sqrt{4n - 3})}{4}$.

- 2. Let P(x) and Q(x) be polynomials with integer coefficients with P(x) being monic. Prove that there exists a polynomial R(x) with integer coefficients such that P(x) divides R(Q(x)).
- 3. Find, with proof, the least odd integer a > 5 for which there exists positive integers m_1, m_2, n_1, n_2 such that

$$a = m_1^2 + n_1^2$$
, $a^2 = m_2^2 + n_2^2$, and $m_1 - n_1 = m_2 - n_2$ is divisible by 3.



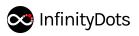
Thailand Team Selection Test for IMO 2012 IPST, Bangkok 5 February 2012

Day 4 Time: 4.5 hours

1. Let a, b, c be non-negative real numbers such that $a + b + c \leq 3$. Prove that

$$\frac{1}{4}a^{2}b(b\sqrt{c}+3)^{2} + \frac{1}{4}b^{2}c(c\sqrt{a}+3)^{2} + \frac{1}{4}c^{2}a(a\sqrt{b}+3)^{2} \\ \leqslant abc(ab^{2}+bc^{2}+ca^{2}) + 3(a^{2}b+b^{2}c+c^{2}a) \leqslant 12$$

- 2. a) Find a continuous, strictly decreasing function $f : \mathbb{R} \to \mathbb{R}$ such that f(f(x)) = 2012x + 2555 for all real numbers x.
 - b) Find all continuous, strictly decreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that f(f(x)) = x + 2555 for all real numbers x.
- 3. Let $\triangle ABC$ be an acute-angled triangle with circumcircle $\odot O$ centered at O. Let P be a point inside $\triangle ABC$ and let lines AP, BP, CP intersect $\odot O$ again at A', B', C', respectively. Let B_c, C_b, A_b, B_a, C_a and A_c , respectively, denote the circumcenters of $\triangle PBC', \triangle PCB', \triangle PAB', \triangle PBA', \triangle PCA'$ and $\triangle PAC'$. Prove that lines B_cC_b, A_bB_a and C_aA_c concur at the midpoint of segment OP.



Thailand Team Selection Test for IMO 2012 IPST, Bangkok 28 March 2012

Day 5 Time: 4.5 hours

1. Does there exist a convex equilateral *n*-gon whose vertices all lie on the curve $y = x^2$, where

a) n = 2012;

b) n = 2013?

2. Determine all pairs (f,g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

3. Let p be an odd prime number. For every integer a, define the number

$$S_a = \frac{a}{1} + \frac{a^2}{2} + \dots + \frac{a^{p-1}}{p-1}.$$

Let m and n be integers such that

$$S_3 + S_4 - 3S_2 = \frac{m}{n}.$$

Prove that p divides m.



Thailand Team Selection Test for IMO 2012 IPST, Bangkok 31 March 2012

Day 6 Time: 4.5 hours

1. Let a, b and c be positive real numbers satisfying $\min(a + b, b + c, c + a) > \sqrt{2}$ and $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a}{(b+c-a)^2}+\frac{b}{(c+a-b)^2}+\frac{c}{(a+b-c)^2}\geqslant \frac{3}{(abc)^2}$$

- 2. Let n > 2 be a positive integer and let $\Phi_n(x)$ be the n^{th} cyclotomic polynomial.
 - a) Prove that $\Phi_n(a) > 0$ for all real numbers a.
 - b) Find, with proof, the values of $\Phi_n(1)$ and $\Phi_n(-1)$.
- 3. Let ABC be an acute triangle with circumcircle Ω . Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

6 of 11



Thailand Team Selection Test for IMO 2012 IPST, Bangkok 4 April 2012

Day 7 Time: 4.5 hours

- 1. For any integer d > 0, let f(d) be the smallest possible integer that has exactly d positive divisors (so for example we have f(1) = 1, f(5) = 16, and f(6) = 12). Prove that for every integer $k \ge 0$ the number $f(2^k)$ divides $f(2^{k+1})$.
- 2. Let G be a graph with vertex set V and clique number $\omega(G) = 100$. Prove that there is a partition of V into sets V_1 and V_2 such that $\omega(G[V_1]) = \omega(G[V_2]) = 50$, where $G[V_1]$ and $G[V_2]$ are the subgraphs of G induced by V_1 and V_2 respectively.¹

Note: the clique number of a graph is the size of its largest clique.

3. Let ABC be a triangle with incentre I and circumcircle ω . Let D and E be the second intersection points of ω with AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and B meet at a point K. Prove that the three lines AE, BD and KP are either parallel or concurrent.





Thailand Team Selection Test for IMO 2012 IPST, Bangkok 5 April 2012

Day 8 Time: 4.5 hours

- 1. Prove that there exists 1000 consecutive positive integers containing exactly 25 primes.
- 2. Let \Im be a convex figure. Prove that there is a point C in \Im such that for any chord AB of \Im through C, we have

$$|AC| \leqslant \frac{2}{3}|AB|.$$

Note: a chord of a convex figure is a line segment with endpoints on the boundary of the figure.

3. Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f(n)\ldots))}_k$.



Thailand Team Selection Test for IMO 2012 IPST, Bangkok 19 April 2012

Day 9 Time: 3 hours

- 1. A positive integer n is given. Prove that there are infinitely many monic polynomials of degree n with integer coefficients whose zeros all lie in the unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. Furthermore, show that all zeros of such polynomials have modulus zero or one.
- 2. Let a and b be positive integers. Suppose that the cyclotomic polynomials Φ_a and Φ_b satisfy

$$gcd(\Phi_a(t), \Phi_b(t)) > 1$$

for some positive integer k. Prove that $\frac{a}{b}$ is a prime power, that is, $\frac{a}{b} = p^k$ for a prime number p and an integer k.



Thailand Team Selection Test for IMO 2012 IPST, Bangkok 26 April 2012

Day 10 Time: 4.5 hours

1. Let a, b, c be positive real numbers. Prove that

$$\frac{a\sqrt{b^3+c^3}}{b^2+c^2} + \frac{b\sqrt{c^3+a^3}}{c^2+a^2} + \frac{c\sqrt{a^3+b^3}}{a^2+b^2} \geqslant \frac{1}{\sqrt{2\left(\frac{a^5+b^5}{ab(a+b)} + \frac{b^5+c^5}{bc(b+c)} + \frac{c^5+a^5}{ca(c+a)}\right)}}.$$

2. For each positive integer k, let t(k) be the largest odd divisor of k. Determine all positive integers a for which there exists a positive integer n, such that all the differences

$$t(n+a) - t(n); t(n+a+1) - t(n+1), \dots, t(n+2a-1) - t(n+a-1)$$

are divisible by 4.

- 3. Let \Im be a collection of subsets of $\{1, 2, \ldots, n\}$ such that
 - (i) every element of \Im is a 3-element set, and
 - (ii) the intersection of any two elements of \Im contains at most one element.

Let f denote the maximum number of elements that \Im can have. Prove that

$$n^2 - 4n \leqslant 6f \leqslant n^2 - n.$$



Day 11 Time: 4.5 hours

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y)f(x-y) = (x-y)g(x+y)$$

for all real numbers x and y.

- 2. Let ABC be a triangle with AB = AC and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B and C at the point E inside the triangle ABC. The line BD intersects the circle through A, E and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incentre of triangle KAB.
- 3. On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of k?

