Thailand Team Selection Test for IMO 2011
IPST, Bangkok
25 December 2010

Day 1
Time: 4 hours

1. Find all pairs of positive integers (m, n) for which it is possible to paint each unit square of an $m \times n$ table either black or white satisfying the following condition: for any unit square U in the table, the number of unit squares V such that U and V have a vertex in common, and U and V are painted with the same color, is odd.

2. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation
$$|f(x + y)| = |f(x) + f(y)|$$
for all real numbers x and y. Show that f also satisfies the functional equation
$$f(x + y) = f(x) + f(y)$$
for all real numbers x and y.

3. Let $\triangle ABC$ be an acute-angled triangle with $AB > AC$, and let AD be its altitude from A to BC. Let the circle Ω with diameter BC intersect AC at E, and let the tangents to Ω at B and E meet at X. Suppose that the lines AX and BC meet at Y. Prove that
$$\frac{1}{DC} - \frac{1}{BD} = \frac{2}{DY}.$$

Thailand Team Selection Test for IMO 2011
IPST, Bangkok
26 December 2010

Day 2
Time: 4 hours

1. Let $f(x)$ and $g(x)$ be polynomials with rational coefficients whose product $f(x)g(x)$ has integer coefficients. Prove that the product of any coefficient of $f(x)$ and any coefficient of $g(x)$ is an integer.

2. Let p and q be odd primes and m a positive integer such that $\nu_q(p-1) = m$. Prove that for any integer $n \geqslant 0$, the order of p modulo q^{m+n} is q^n and $\nu_q\left(p^{q^n} - 1\right) = m + n$.

   Note: as usual, $\nu_q(x)$ is the largest integer k such that $q^k \mid x$.

3. Let $\triangle ABC$ be a triangle with $AB \leqslant AC$ and let P be an interior point lying on the angle bisector of $\angle BAC$. Let D and E be points on segments PC and PB, respectively, so that $\angle PBD = \angle PCE$. Lines BD and AC intersect at X, and lines CE and AB intersect at Y. Prove that $BX \leqslant CY$.

Thailand Team Selection Test for IMO 2011
IPST, Bangkok
22 January 2011

Day 3
Time: 4.5 hours

1. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\sqrt[3]{\frac{abc}{(a+3)(b+3)(c+3)}} \leqslant \frac{9}{32}\left(\left(\frac{3-a}{1+2bc}\right)^3 + \left(\frac{3-b}{1+2ca}\right)^3 + \left(\frac{3-c}{1+2ab}\right)^3\right).$$

2. Let $\triangle ABC$ be an acute-angled triangle with orthocenter H and nine-point center N. Consider points Y and Z on sides CA and CB respectively such that $\measuredangle(AC, HY) = -\frac{\pi}{3}$ and $\measuredangle(AB, HZ) = \frac{\pi}{3}$ as directed angles. Let U be the circumcenter of $\triangle HYZ$. Prove that A, N, U are collinear.[1]

3. A graph with $2n$ vertices and $2n(n-1)$ edges is given, where $n > 1$ is an integer. Prove that it is possible to color some vertices and edges of the graph red so that each both endpoints of each red edge are red, and each red vertex belongs to exactly n red edges.

[1] This problem was given again on TST 2016.

Thailand Team Selection Test for IMO 2011
IPST, Bangkok
23 January 2011

Day 4
Time: 4.5 hours

1. A 16×16 matrix whose elements are $+1$ or -1 is given. Suppose that for any two distinct columns, the sum of the products of the elements in the same row is nonpositive. Show that the number of $+1$'s in the matrix is at most 160.

2. Determine all prime numbers p for which there exist integers a and b such that

$$p^2 = a^2 + b^2 \quad \text{and} \quad p \mid a^3 + b^3 - 4.$$

3. A triangle $\triangle ABC$ is given, with circumcircle Γ and an incircle of unit radius. Let Γ_A be the circle tangent to the lines AB and AC and the circle Γ internally. Define Γ_B and Γ_C similarly. Let R_A, R_B and R_C be the radii of Γ_A, Γ_B and Γ_C, respectively. Prove that $R_A + R_B + R_C \geqslant 4$.

Thailand Team Selection Test for IMO 2011
IPST, Bangkok
14 March 2011

Day 5
Time: 4.5 hours

1. Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

2. Is it possible to color all rational numbers with one of two colors such that if x and y are rational numbers such that $x \neq y$ and $xy = 1$ or $x + y \in \{0, 1\}$, then x and y must be different colors?

3. A sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k, x_{2k-1} = (-1)^{k+1}x_k$ for all $k \geqslant 1$. Prove that $x_1 + x_2 + \ldots + x_n \geqslant 0$ for all $n \geqslant 1$.

Thailand Team Selection Test for IMO 2011
IPST, Bangkok
15 March 2011

Day 6
Time: 4.5 hours

1. An 8×8 square is divided into 64 unit squares. In some of the unit squares a diagonal is drawn so that no two diagonals share a common point (including endpoints). Find the maximum possible number of diagonals that could be drawn under this condition.

2. A circle Ω through the vertex A of $\triangle ABC$ is tangent to segment BC at D. Let E be the projection from D onto AC, and let F be a point on Ω lying on the opposite side of D with respect to line EC so that $\angle EFC = 2\angle ECD$. Suppose that CF meets Ω again at G. Prove that AG is parallel to BC.

3. Find the smallest number n such that there exist polynomials $f_1, f_2, \ldots, f_n$ with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \ldots + f_n(x)^2.$$

Thailand Team Selection Test for IMO 2011
IPST, Bangkok
23 March 2011

Day 7
Time: 4.5 hours

1. Let a, b, c be real numbers with $ab + bc + ca = 1$. Prove that
$$\frac{(a+b)^2+1}{c^2+2} + \frac{(b+c)^2+1}{a^2+2} + \frac{(c+a)^2+1}{b^2+2} \geqslant 3.$$

2. Determine the least positive integer n for which there exists a set $\{s_1, s_2, \ldots, s_n\}$ consisting of n distinct positive integers such that
$$\left(1 - \frac{1}{s_1}\right)\left(1 - \frac{1}{s_2}\right)\cdots\left(1 - \frac{1}{s_n}\right) = \frac{51}{2010}.$$

3. The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of an acute-angled triangle ABC. Prove that the incenter of triangle ABC lies inside triangle XYZ.

Thailand Team Selection Test for IMO 2011
IPST, Bangkok
24 March 2011

Day 8
Time: 4.5 hours

1. Let D and E be interior points of sides AB and AC of a triangle ABC, respectively, so that the line DE intersects the extension of BC at F. Prove that $\triangle ADE$ is similar to the triangle whose vertices are the circumcenters of $\triangle ABC, \triangle DFB$ and $\triangle EFC$.

2. A tourist preparing to visit Dreamland finds out that

   (i) there are 1024 cities in Dreamland, each labeled with a distinct integer from the set $\{0, 1, 2, \ldots, 1023\}$;

   (ii) two cities numbered m and n are connected by a single road if and only if the binary expansions of m and n differ in exactly one digit; and

   (iii) during his visit, exactly 8 roads will be closed for repairing.

   Prove that the tourist can organize a closed path passing through every city exactly once, while only traveling through the roads that remain open.

3. Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:

$$f\left(f(x)^2 y\right) = x^3 f(xy).$$

Thailand Team Selection Test for IMO 2011
IPST, Bangkok
30 March 2011

Day 9
Time: 4.5 hours

1. Let $a, b, c > 0$. Prove that

$$\frac{(a+b)^2}{c(a+b+2c)} + \frac{(b+c)^2}{a(b+c+2a)} + \frac{(c+a)^2}{b(c+a+2b)} \geqslant \sqrt{\frac{3a(b+c)}{b^2+4bc+c^2}} + \sqrt{\frac{3b(c+a)}{c^2+4ca+a^2}} + \sqrt{\frac{3c(a+b)}{a^2+4ab+b^2}}.$$

2. For each integer $n \geqslant 2$, denote by A_n the set of solutions of the equation

$$x = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \cdots + \left\lfloor \frac{x}{n} \right\rfloor .$$

Prove that $A := \bigcup_{n\geqslant 2} A_n$ is finite, and find $\max A$.

3. Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90^\circ$, prove that $2\angle BDA = \angle CDE$.

Thailand Team Selection Test for IMO 2011
IPST, Bangkok
31 March 2011

Day 10
Time: 4.5 hours

1. Consider the recurrence relation $K_0 = 1$;
$$K_n = 1 + \min\left\{2K_{\lfloor \frac{n}{2} \rfloor}, 3K_{\lfloor \frac{n}{3} \rfloor}\right\}$$
for $n \geqslant 0$. Prove or disprove the following statement:
$$K_n \geqslant n \text{ for all integers } n \geqslant 0.$$

2. A finite set of $n \geqslant 4$ points is given so that the distance between any two of the n points is an integer. Prove that at least $\frac{1}{6}$ of these distances are divisible by 3.

3. $n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let w_i and l_i be respectively the number of wins and losses of the i-th player. Prove that
$$\sum_{i=1}^{n} (w_i - l_i)^3 \geqslant 0.$$

Thailand Team Selection Test for IMO 2011
IPST, Bangkok
17 April 2011

Day 11
Time: 4.5 hours

1. Let the real numbers a, b, c, d satisfy the relations $a+b+c+d = 6$ and $a^2+b^2+c^2+d^2 = 12$. Prove that
$$36 \leqslant 4\left(a^3+b^3+c^3+d^3\right) - \left(a^4+b^4+c^4+d^4\right) \leqslant 48.$$

2. Let $A_1A_2 \ldots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections $P_1, \ldots, P_n$ onto lines $A_1A_2, \ldots, A_nA_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1, \ldots, X_n$ on sides $A_1A_2, \ldots, A_nA_1$ respectively,
$$\max\left\{\frac{X_1X_2}{P_1P_2}, \ldots, \frac{X_nX_1}{P_nP_1}\right\} \geqslant 1.$$

3. Let a, b be integers, and let $P(x) = ax^3 + bx$. For any positive integer n we say that the pair (a, b) is *n-good* if $n|P(m) - P(k)$ implies $n|m-k$ for all integers m, k. We say that (a, b) is *very good* if (a, b) is n-good for infinitely many positive integers n.

   a) Find a pair (a, b) which is 51-good, but not very good.

   b) Show that all 2010-good pairs are very good.

Thailand Team Selection Test for IMO 2011
IPST, Bangkok
18 April 2011

Day 12
Time: 4.5 hours

1. Let I be the incenter of $\triangle ABC$ whose incircle touches sides AB, BC and CA at points X, Y and Z, respectively. Let H' be the orthocenter of $\triangle XYZ$. Suppose that lines IH' and BC intersect at D and lines AD and XH' intersect at P. Prove that $PY = YX$.

2. On some planet, there are 2^N countries $(N \geqslant 4.)$ Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is diverse if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

3. Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer n.

Thailand Team Selection Test for IMO 2011
IPST, Bangkok
29 April 2011

Day 13
Time: 4.5 hours

1. Let a, b, c be positive real numbers. Prove that
$$\frac{4a^4}{(a+b)(a^2+b^2)} + \frac{4b^4}{(b+c)(b^2+c^2)} + \frac{4c^4}{(c+a)(c^2+a^2)} \geqslant \frac{5b^3-a^3}{ab+3b^2} + \frac{5c^3-b^3}{bc+3c^2} + \frac{5a^3-c^3}{ca+3a^2}.$$

2. Let I be the incenter of $\triangle ABC$, and let ℓ be a tangent line to the incircle of $\triangle ABC$, which is not one of its sides. The line ℓ meets the sides AB, BC and the extension of CA at points X, Y and Z respectively. Let AY intersect CX at P, and let lines IP and BZ intersect at Q. Prove that if A, C, Y, X are concyclic, then $ZI^2 = ZQ \cdot ZB$.

3. Let $n \geqslant 2$ be an integer. A group of people is called *n-compact* if for every person in the group, there are n people different from him/her who are acquainted with one another. Find the maximum possible integer N such that every n-compact group of N people contains a subgroup of $n+1$ people, all of whom are acquainted with one another.

Thailand Team Selection Test for IMO 2011
IPST, Bangkok
30 April 2011

Day 14
Time: 4.5 hours

1. Let $f : (0, \infty) \to \mathbb{R}$ satisfy the functional equation

$$f(x+y) = f\left(\frac{x+y}{xy}\right) f(xy)$$

for all $x, y \in (0, \infty)$. Show that f also satisfies the functional equation

$$f(xy) = f(x) + f(y)$$

for all $x, y \in (0, \infty)$.

2. Let a and n be two positive integers such that the prime factors of a are all greater than n. Prove that $n!$ divides $(a-1)(a^2-1)\cdots(a^n-1)$.

3. Let A_1, A_2, A_3, A_4 be four points in the plane, chosen so that A_4 is the centroid of $\triangle A_1A_2A_3$. Find a point A_5 on the plane that maximizes the ratio[2]

$$\frac{\min\limits_{1\leqslant i<j<k\leqslant 5} A_iA_jA_k}{\max\limits_{1\leqslant i<j<k\leqslant 5} A_iA_jA_k}.$$

[2]Presumably this is the ratio of areas?