16th Thailand Mathematical Olympiad
Silpakorn University, Nakhon Pathom
19 May 2019

Day 1
Time: 4.5 hours

1. Let $ABCDE$ be a convex pentagon such that $\angle AEB = \angle BDC = 90^\circ$ and AC bisects both $\angle BAE$ and $\angle DCB$. The circumcircle of $\triangle ABE$ intersects line AC again at P.

   a) Show that P is the circumcenter of $\triangle BDE$.

   b) Show that points A, C, D, E lie on a circle.

2. Let a and b be distinct positive integers. If a and b are coprime, show that

$$\frac{2a(a^2+b^2)}{a^2-b^2}$$

   is not an integer.

3. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f\big(x + yf(x) + y^2\big) = f(x) + 2y$$

   for all positive reals x, y.

4. A rabbit initially stands on the point 0 and repeatedly jumps on the real line. In each jump, the rabbit can jump to any integral point except the one it is currently standing on. Let $N(a)$ be the number of sequences of jumps with a total distance of 2019 which ends at point a. Find all integers a such that $N(a)$ is odd.

5. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{4a-1}{(2b+1)^2} + \frac{4b-1}{(2c+1)^2} + \frac{4c-1}{(2a+1)^2} \geqslant 1.$$

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6. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xf(y) + yf(x) \leqslant xy$$

for all real numbers x, y.

7. Let $A = \{-2562, -2561, \ldots, 2561, 2562\}$. Show that, given a bijective function $f : A \to A$,

$\sum_{k=1}^{2562} |f(k) - f(-k)|$ is maximized if and only if $f(k)f(-k) < 0$ for all $k = 1, 2, \ldots, 2562$.

8. In a triangle $\triangle ABC$ with $AB \neq AC$, the incircle centered at I is tangent to BC at D. The circumcircle of $\triangle ABC$ intersects the circle with diameter AI again at K, and intersects the line AI again at M. Prove that the points K, D, M lie on a line.

9. A *chaisri* figure is a triangle formed by three vertices are vertices of a regular 2019-gon. Different chaisri figures may be formed by different regular 2019-gons.

   A *thubkaew* figure is a convex polygon which can be dissected into multiple *chaisri* figures, where each vertex of a dissected chaisri figure may not necessarily lie on the border of the convex polygon. Determine the maximum number of vertices a thubkaew figure may have.

10. Show that there are infinitely many odd integers n such that $n! + 1$ is not a prime number.