15th Thailand Mathematical Olympiad - Unofficial Solutions

1. From simple angle chasing, we get $\angle BPQ + \angle BCQ = (90 + \frac{A+B}{2}) + \frac{C}{2} = 180°$ Therefore, B, C, P, Q are on the same circle. From $\angle QBP = \angle QCP$ and $\angle BPD = \angle DQC$, we get $\angle PSQ = \angle PRQ$. Therefore, P, Q, R, S are on the same circle.

2. Suppose that there is a function f. Let $P(x, y)$ denote the statement $f(x + f(y)) = f(x) + y^2$. $P(x, 0)$ gives $f(x + f(0)) = f(x)$ so $f(f(0)) = f(0)$. Comparing $P(x, 0)$ and $P(x, f(0))$ gives $f(0)^2 = 0$, therefore $f(0) = 0$.

   Now look at $P(0, x)$: $f(f(x)) = x^2$, $P(0, f(x))$: $f(x^2) = f(x)^2$, and $P(f(x), x)$: $f(2f(x)) = 2x^2$

   Substituting x with $f(x)$ in the last equation gives $f(2x^2) = 2f(x)^2$, while $P(f(x), x)$ gives $f(f(2f(x))) = f(2x^2)$, so
   $$4f(x)^2 = 2f(x)^2$$
   for all real x, so $f(x) \equiv 0$. However this is clearly not a solution, which is the desired contradiction.

   REMARK: There is also a solution by looking at $f(\mathbb{R})$.

3. Suppose there are two servants A and B such that the sum of capacities of A's flash drives is equal to of B's. (Else the problem is already done.) It's easy to see that A and B must have the exact same set of flash drives, say, with capacities $\{x, y, z\}$. Now pick a capacity $w \notin \{x, y, z\}$, and consider the servant that is not A or B, and does not have the w-flash drive. This servant must have at least one of the flash drives with capacities $\{x, y, z\}$, say the x-flash drive. We can now choose w and x as the two capacities.

4. ANSWER: 4/27.

   Plugging $c = -a - b$ in the problem, we want to find the maximum value of
   $$\frac{a^2b^2(a+b)^2}{(a^2+ab+b^2)^3}.$$
   From
   $$\begin{aligned}\frac{4}{27} - \frac{a^2b^2(a+b)^2}{(a^2+ab+b^2)^3} &= \frac{4(a^2+ab+b^2)^3 - 27a^2b^2(a+b)^2}{27(a^2+ab+b^2)^3} \\ &= \frac{(a-b)^2(2a+b)^2(2b+a)^2}{27(a^2+ab+b^2)^3} \\ &\geqslant 0,\end{aligned}$$
   it follows that the maximum value of $\frac{a^2b^2c^2}{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)}$ is $\frac{4}{27}$ and this holds when (a, b, c) is $(x, x, -2x)$ and its permutations.

5. ANSWER: 625.

   From $x^5 \equiv x \pmod 5$, it follows that $5 \mid a + b$, where it's easy to show that $5 \mid a^4 - a^3b + a^2b^2 - ab^3 + b^4$ but $25 \nmid a^4 - a^3b + a^2b^2 - ab^3 + b^4$. Therefore, $5^4 \mid a + b$. Hence, the minimum value of $a + b$ is 625 which is attained at, for example, $(a, b) = (1, 624)$.

   REMARK: This problem is basically the LIFTING THE EXPONENT lemma.

6. a) It suffices to show that 2 and 3 divides each of x, y, z separately.

   To show that 2 divides each of x, y, z, first $3y^3 = 2(2z^4 - x^2)$ implies $2 \mid y$, so $y = 2y_1$ for some $y_1 \in \mathbb{Z}_{\geqslant 0}$. Then, $x^2 = 2(6y_1^3 + z^4)$ implies $2 \mid x$, hence $x = 2x_1$ for some $x' \in \mathbb{Z}_{\geqslant 0}$. Finally, this implies $z^4 = 2n^2 - 6m^3$ so $2 \mid z$.

To show that 3 divides each of x, y, z, note that we have $4z^4 \equiv 2x^2 \pmod 3$, but $4z^4 \mod 3 \in \{0,1\}$ and $2x^2 \mod 3 \in \{0,2\}$, so $2x^2 \equiv 4z^4 \equiv 0 \mod 3$, hence $3 \mid x, z$. Now let $x = 3x_2, z = 3z_2$ where $x_2, z_3 \in \mathbb{Z}_{\geqslant 0}$ to get $y^3 = 3(36z_2^4 - x_2^2)$, so $3 \mid y$.

b) $(144t^6, 24t^3, 12t^3) \in A$ for all positive integers t.

7. ANSWER: 119.

Let x_i be the number of $a \in S$ that is colored with the i^{th} color. It's easy to see that $m = \sum_{i=1}^{25}(2^{x_i} - 1)$.

If there are $i, j \in \{1, 2, ..., 25\}$ such that $x_i - x_j \geqslant 2$, then the adjustment $(x_i, x_j) \to (x_i - 1, x_j + 1)$ decreases the value of m, hence when m is minimal, $|x_i - x_j| \leqslant 1$ for all i, j, which forces the multiset $\{x_1, \ldots, x_{25}\}$ to contain 11 threes and 14 twos.

Hence $m \geqslant 11 \cdot (2^3 - 1) + 14 \cdot (2^2 - 1) = 119$, and this lower bound is obviously attainable.

8. ANSWER: 10.

First, note that the set of ticket numbers $\{101, 102, ..., 121\}$ shows that 10 is attainable. Now we'll show that $n \leqslant 10$. Let the ticket numbers be $x_1 < x_2 < ... < x_{2n+1}$

From the problem, we get

$$x_1 + ... + x_{2n+1} > 2330 \tag{1}$$
$$x_{n+2} + ... + x_{2n+1} \leqslant 1165 \tag{2}$$

So, $x_1 + ... + x_{n+1} > x_{n+2} + ... + x_{2n+1}$, which implies

$$x_1 > \sum_{i=1}^{n}(x_{n+1+i} - x_{i+1}) \geqslant \sum_{i=1}^{n} n.$$

Therefore $x_1 \geqslant n^2 + 1$, and it's easy to show that $x_i \geqslant n^2 + i$ for all 1. From (2), we get $1165 \geqslant \sum_{i=1}^{n} n^2 + n + 1 + i$, which is $1165 \geqslant \frac{2n^3+3n^2+3n}{2}$, so $n \leqslant 10$.

9. Let the incircles of $\triangle ABP$ and $\triangle ACP$ be tangent to BC at M and N respectively. It's easy to see that $\angle LPK = 90°$. Let $\angle APK = x, \angle LPA = 90° - x$. Let O be the midpoint of KL, so O is the circumcenter of $KPLQ$. By Power of Point, we get $AQ \cdot AP = AO^2 - OK^2$.

Applying law of cosine to $\triangle AOK$ and $\triangle BOK$ gives

$$AK^2 + AL^2 = 2(AO^2 + OK^2).$$

Applying law of cosine to $\triangle APK$ and $\triangle APL$, we get

$$AK^2 = AP^2 + PK^2 - 2AP \cdot PK \cdot \cos x$$
$$AL^2 = AP^2 + PL^2 - 2AP \cdot PL \cdot \sin x$$

Adding the two equations gives

$$AK^2 + AL^2 = 2AP^2 + PK^2 + PL^2 - 2AP(PK \cdot \cos x + PL \cdot \sin x).$$

From $\angle LPK = 90°$, we get $PK^2 + PL^2 = 4OK^2$. Therefore,

$$\begin{aligned} 2(AO^2 + OK^2) &= 2AP^2 + 4OK^2 - 2AP(PK \cdot \cos x + PL \cdot \sin x) \\ AO^2 - OK^2 &= AP(AP - PK \cos x + PL \cdot \sin x) \\ AP \cdot AQ &= AP(AP - PK \cos x + PL \cdot \sin x) \end{aligned}$$

Finally, from $PK \cdot \cos x = PM = \frac{AP+PB-AB}{2}$ and $PL \cdot \sin x = PN = \frac{AP+PC-AC}{2}$,

$$AQ = \frac{AB + AC - BP - CP}{2} = \frac{AB + AC - BC}{2} = AD.$$

10. We show that the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(k) = \frac{1}{c}\big(g(y_k + 2k) - 2g(y_k + k) + g(y_k)\big),$$

where $y_k = 2019 + 2|k|$ works. Fix a $k \in \mathbb{R}$ and note that $y_k, y_k + k, y_k + 2k > 2018$. Comparing $(x, y) = (x, y_k)$ and $(x + k, y_k + k)$ in the original equation gives

$$a\big(f(x + y_k + 2k) - f(x + y_k)\big) = c\big(f(x + k) - f(x)\big) + g(y_k + k) - g(y_k). \tag{3}$$

Substituting $(x, y_k) \to (x - k, y_k + k)$ in (3) gives

$$a\big(f(x + y_k + 2k) - f(x + y_k)\big) = c\big(f(x) - f(x - k)\big) + g(y_k + 2k) - g(y_k + k).$$

Therefore, for all $x \in \mathbb{R}$,

$$f(x + k) + f(x - k) - 2f(x) = \frac{1}{c}\big(g(y_k + 2k) - 2g(y_k + k) + g(y_k)\big) = h(k),$$

hence h satisfy the required condition.