15th Thailand Mathematical Olympiad
Ratchasima Wittayalai School, Nakhon Ratchasima
6 May 2018

Day 1
Time: 4.5 hours

1. In $\triangle ABC$, the incircle is tangent to the sides BC, CA, AB at D, E, F respectively. Let P and Q be the midpoints of DF and DE respectively. Lines PC and DE intersect at R, and lines BQ and DF intersect at S. Prove that

   a) Points B, C, P, Q lie on a circle.

   b) Points P, Q, R, S lie on a circle.

2. Show that there are no functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + f(y)) = f(x) + y^2$$

   for all real numbers x and y.

3. Karakade has three flash drives of each of the six capacities $1, 2, 4, 8, 16, 32$ gigabytes. She gives each of her 6 servants three flash drives of different capacities.

   Prove that either there are two capacities where each servant has at most one of the two capacities, or all servants have flash drives with different sums of capacities.

4. Let a, b, c be nonzero real numbers such that $a + b + c = 0$. Determine the maximum possible value of

$$\frac{a^2b^2c^2}{(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2)}.$$

5. Let a, b be positive integers such that $5 \nmid a, b$ and $5^5 \mid a^5 + b^5$. What is the minimum possible value of $a + b$?

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Day 2
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6. Let A be the set of all triples (x, y, z) of positive integers satisfying $2x^2 + 3y^3 = 4z^4$.

   a) Show that if $(x, y, z) \in A$ then 6 divides all of x, y, z.

   b) Show that A is an infinite set.

7. We color each number in the set $S = \{1, 2, ..., 61\}$ with one of 25 given colors, where it is not necessary that every color gets used. Let m be the number of non-empty subsets of S such that every number in the subset has the same color. What is the minimum possible value of m?

8. There are $2n + 1$ tickets, each with a unique positive integer as the ticket number. It is known that the sum of all ticket numbers is more than 2330, but the sum of any n ticket numbers is at most 1165. What is the maximum value of n?

9. In $\triangle ABC$ the incircle is tangent to AB at D. Let P be a point on BC different from B and C, and let K and L be incenters of $\triangle ABP$ and $\triangle ACP$ respectively. Suppose that the circumcircle of $\triangle KPL$ cuts AP again at Q. Prove that $AD = AQ$.

10. Let a, b, c be nonzero real numbers. Suppose that functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy

$$af(x + y) + bf(x - y) = cf(x) + g(y)$$

for all real numbers x and y such that $y > 2018$. Show that there exists a function $h : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) + f(x - y) = 2f(x) + h(y)$$

for all real numbers x and y.