14th Thailand Mathematical Olympiad
Walailak University, Nakhon Si Thammarat
11 May 2017

Day 1
Time: 4.5 hours

1. Let p be a prime. Show that $\sqrt[3]{p} + \sqrt[3]{p^5}$ is irrational.

2. A cyclic quadrilateral $\square ABCD$ has circumcenter O; its diagonals AC and BD intersect at G. Let P, Q, R, S be the circumcenters of $\triangle AGB, \triangle BGC, \triangle CGD, \triangle DGA$ respectively. Lines PR and QS intersect at M. Show that M is the midpoint of OG.

3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) - y) \leqslant xf(x) + f(y)$$

for all real numbers x, y.

4. In a math competition, 14 schools participate, each sending 14 students. The students are separated into 14 groups of 14 so that no two students from the same school are in the same group.

The tournament organizers noted that, from the competitors, exactly 15 have participated in the competition before. The organizers want to select two representatives, with the conditions that they must be former participants, must come from different schools, and must also be in different groups. It turns out that there are n ways to do this. What is the minimum possible value of n?

5. Does there exist 2017 consecutive positive integers, none of which could be written as $a^2 + b^2$ for some integers a, b? Justify your answer.

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6. In an acute triangle $\triangle ABC$, D is the foot of altitude from A to BC. Suppose that $AD = CD$, and define N as the intersection of the median CM and the line AD. Prove that $\triangle ABC$ is isosceles if and only if $CN = 2AM$.

7. Show that no pairs of integers (m, n) satisfy $2560m^2 + 5m + 6 = n^5$.

8. Let a, b, c be side lengths of a right triangle. Determine the minimum possible value of
$$\frac{a^3 + b^3 + c^3}{abc}.$$

9. Determine all functions f on the set of positive rational numbers such that
$$f(xf(x) + f(y)) = f(x)^2 + y$$
for all positive rational numbers x, y.

10. A *lattice point* is defined as a point on the plane with integer coordinates. Show that for all positive integers n, there is a circle on the plane with exactly n lattice points in its interior (not including its boundary).