

13th Thailand Mathematical Olympiad
Prince of Songkhla University, Pattani
17 May 2016

Day 1

Time: 4.5 hours

1. Let ABC be a triangle with $AB \neq AC$. Let the angle bisector of $\angle BAC$ intersect BC at P and intersect the perpendicular bisector of segment BC at Q . Prove that

$$\frac{PQ}{AQ} = \left(\frac{BC}{AB + AC} \right)^2.$$

2. Let M be a positive integer, and $A = \{1, 2, \dots, M + 1\}$. Show that if f is a bijection from A to A then

$$\sum_{n=1}^M \frac{1}{f(n) + f(n+1)} > \frac{M}{M+3}.$$

3. Determine all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$f(f(x)f(y) + f(y)f(z) + f(z)f(x)) = f(x) + f(y) + f(z)$$

for all real numbers x, y, z .

4. Each point on the plane is colored either red, green, or blue. Prove that there exists an isosceles triangle whose vertices all have the same color.
5. Let p_1, p_2, \dots be the sequence of positive integers defined by $p_1 = 2$, and for all positive integers n , p_{n+1} is the smallest prime divisor of $np_1^{1!}p_2^{2!} \cdots p_n^{n!} + 1$. Prove that all prime numbers appear in this sequence.

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Day 2

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6. Let m and n be positive integers. Prove that if $m^{4^n+1} - 1$ is a prime number then there exists an integer $t \geq 0$ such that $n = 2^t$.
7. Let $P(x) = a_{2016}x^{2016} + a_{2015}x^{2015} + \cdots + a_1x + a_0$ be a degree 2016 polynomial, where its coefficients satisfy

$$|a_1 + a_3 + \cdots + a_{2015}| > |a_0 + a_2 + \cdots + a_{2016}|.$$

Prove that the number of roots of $P(x)$ with absolute value less than 1 is odd, where roots are counted with multiplicity.

8. Let $\triangle ABC$ be an acute triangle with incenter I . The line through I parallel to AC intersects AB at M , and the line through I parallel to AB intersects AC at N . Let the line MN intersect the circumcircle of $\triangle ABC$ at X and Y . Let Z be the midpoint of arc BC (not containing A). Prove that I is the orthocenter of $\triangle XYZ$.
9. A real number $a \neq 0$ is given. Determine all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$f(x)f(y) + f(x+y) = axy$$

for all real numbers x, y .

10. A *Pattano coin* is a coin which has a blue side and a yellow side. A positive integer not exceeding 100 is written on each side of every coin (the sides may have different integers). Two Pattano coins are *identical* if the number on the blue side of both coins are equal and the number on the yellow side of both coins are equal. Two Pattano coins are *pairable* if the number on the blue side of both coins are equal or the number on the yellow side of both coins are equal. Given 2559 Pattano coins such that no two coins are identical. Show that at least one Pattano coin is pairable with at least 50 other coins.