10th Thailand Mathematical Olympiad
Burapha University, Chonburi
14 May 2013

Day 1
Time: 4.5 hours

1. Find the largest integer that divides $p^4 - 1$ for all primes $p > 4$.

2. Let $\triangle ABC$ be a triangle with $\angle ABC > \angle BCA \geqslant 30^\circ$. The angle bisectors of $\angle ABC$ and $\angle BCA$ intersect CA and AB at D and E respectively, and BD and CE intersect at P. Suppose that $PD = PE$ and the incircle of $\triangle ABC$ has unit radius. What is the maximum possible length of BC?

3. Each point on the plane is colored either red or blue. Show that there are three points of the same color that form a triangle with side lengths $1, 2, \sqrt{3}$.

4. Determine all monic polynomials $p(x)$ having real coefficients and satisfying the following two conditions.
   - $p(x)$ is nonconstant, and all of its roots are distinct reals
   - If a and b are roots of $p(x)$ then $a + b + ab$ is also a root of $p(x)$.

5. Find a five-digit positive integer n (in base 10) such that $n^3 - 1$ is divisible by 2556 and which minimizes the sum of digits of n.

6. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$(x^2 + y^2)f(xy) = f(x)f(y)f(x^2 + y^2)$$
for all real numbers x, y.

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Day 2
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7. Let $P_1, \ldots, P_{2556}$ be distinct points in a regular hexagon $ABCDEF$ with unit side length. Suppose that no three points in the set $S = \{A, B, C, D, E, F, P_1, \ldots, P_{2556}\}$ are collinear. Show that there is a triangle whose vertices are in S and whose area is less than $\frac{1}{1700}$.

8. Let $p(x) = x^{2013} + a_{2012}x^{2012} + a_{2011}x^{2011} + \cdots + a_1x + a_0$ be a polynomial with real coefficients with roots $-b_{1006}, -b_{1005}, \ldots, -b_1, 0, b_1, \ldots, b_{1005}, b_{1006}$, where $b_1, b_2, \ldots, b_{1006}$ are positive reals with product 1. Show that $a_3a_{2011} \geqslant 1012036$.

9. Let $\square ABCD$ be a convex quadrilateral, and let M and N be midpoints of sides AB and CD respectively. Point P is chosen on CD so that $MP \perp CD$, and point Q is chosen on AB so that $NQ \perp AB$. Show that $AD \parallel BC$ if and only if $\frac{AB}{CD} = \frac{MP}{NQ}$.

10. Find all pairs of positive integers (x, y) such that $\frac{xy^3}{x+y}$ is the cube of a prime.

11. Let m, n be positive integers. There are n piles of gold coins, so that pile i has $a_i > 0$ coins in it ($i = 1, \ldots, n$). Consider the following game:

    Step 1. Nadech picks sets $B_1, B_2, \ldots, B_n$, where each B_i is a nonempty subset of $\{1, 2, \ldots, m\}$, and gives them to Yaya.

    Step 2. Yaya picks a set S which is also a nonempty subset of $\{1, 2, \ldots, m\}$.

    Step 3. For each $i = 1, 2, \ldots, n$, Nadech wins the coins in pile i if $B_i \cap S$ has an even number of elements, and Yaya wins the coins in pile i if $B_i \cap S$ has an odd number of elements.

    Show that, no matter how Nadech picks the sets $B_1, B_2, \ldots, B_n$, Yaya can always pick S so that she ends up with more gold coins than Nadech.

12. Let ω be the incircle of $\triangle ABC$; ω is tangent to sides BC and AC at D and E respectively. The line perpendicular to BC at D intersects ω again at P. Lines AP and BC intersect at M. Let N be a point on segment AC so that $AE = CN$. Line BN intersect ω at Q (closer to B) and intersect AM at R. Show that the area of $\triangle ABR$ is equal to the area of $\square PQMN$.