9th Thailand Mathematical Olympiad
Prince of Songkhla University, Songkhla
15 May 2012

Day 1
Time: 4.5 hours

1. Let $\triangle ABC$ be a right triangle with $\angle B = 90°$. Let P be a point on side BC, and let ω be the circle with diameter CP. Suppose that ω intersects AC and AP again at Q and R, respectively. Show that $CP^2 = AC \cdot CQ - AP \cdot PR$.

2. Let $a_1, a_2, \ldots, a_{2012}$ be pairwise distinct integers. Show that the equation
$$(x - a_1)(x - a_2) \cdots (x - a_{2012}) = (1006!)^2$$
has at most one integral solution.

3. Let $m, n > 1$ be coprime odd integers. Show that
$$\left\lfloor \frac{m^{\phi(n)+1} + n^{\phi(m)+1}}{mn} \right\rfloor$$
is an even integer, where ϕ is Euler's totient function.

4. Let $\square ABCD$ be a unit square. Points E, F, G, H are chosen outside $\square ABCD$ so that $\angle AEB = \angle BFC = \angle CGD = \angle DHA = 90°$. Let O_1, O_2, O_3, O_4, respectively, be the incenters of $\triangle ABE, \angle BCF, \angle CDG, \angle DAH$. Show that the area of $\square O_1O_2O_3O_4$ is at most 1.

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f\big(f(x) + xf(y)\big) = 3f(x) + 4xy$$
for all real numbers x, y.

6. At Hat Yai Witthayalai School, $n \geqslant 100$ freshmen participate in a rock-paper-scissors tournament where each pair of freshmen play each other exactly once. Each win is worth 2 points, each draw 1 point, and each loss 0 points. At the end of the tournament, the total score of each freshman is calculated, and awards are given out as follows:

   - The Math Department wins a computer if and only if for every group of 100 freshmen, there is a freshman who won against all other 99 and another freshman who lost against all other 99.
   - The Math Department wins a printer if and only if the total scores of all freshmen are pairwise distinct.

   Find the least possible value of n for which the following statement is true: if the Math Department wins a computer, then it also wins a printer.

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Day 2
Time: 4.5 hours

7. Let a, b, m be integers such that $\gcd(a, b) = 1$ and $5 \mid ma^2 + b^2$. Show that there exists an integer n such that $5 \mid m - n^2$.

8. $4n$ first grade students at Songkhla Primary School, including $2n$ boys and $2n$ girls, participate in a taekwondo tournament where every pair of students compete against each other exactly once. The tournament is scored as follows:

   - In a match between two boys or between two girls, a win is worth 3 points, a draw 1 point, and a loss 0 points.
   - In a math between a boy and a girl, if the boy wins, he receives 2 points, else he receives 0 points. If the girl wins, she receives 3 points; if she draws, she receives 2 points, and if she loses, she receives 0 points.

   After the tournament, the total score of each student is calculated. Let P be the number of matches ending in a draw, and let Q be the total number of matches. Suppose that the maximum total score is $4n - 1$. Find P/Q.

9. Let n be a positive integer and let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + 1$ be a polynomial with positive real coefficients. Under the assumption that the roots of P are all real, show that $P(x) \geqslant (x + 1)^n$ for all $x > 0$.

10. Let x be an irrational number. Show that there are integers m and n such that

$$\frac{1}{2555} < mx + n < \frac{1}{2012}.$$

11. Let $\triangle ABC$ be an acute triangle, and let P be the foot of altitude from C to AB. Let ω be the circle with diameter BC. The tangents from A to ω are drawn touching ω at D and E. Lines AD and AE intersect line BC at M and N respectively, so that B lies between M and C. Let CP intersect DE at Q, ME intersect ND at R, and let QR intersect BC at S. Show that QS bisects $\angle DSE$.

12. Let a, b, c be positive integers. Show that if $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ is an integer then abc is a perfect cube.