8th Thailand Mathematical Olympiad
Chiang Mai University, Chiang Mai
3 May 2011

Day 1
Time: 4 hours

1. Let $n \geqslant 3$ be a positive integer. Suppose that p and q are primes such that $p \mid n!$ and $q \mid (n-1)! - 1$. Prove that $p < q$.

2. Find all functions f on the set of positive integers satisfying

$$f(2m + 2n) = f(m)f(n)$$

for all positive integers m, n.

3. Given a triangle ABC which $\angle C = 90°$, D is a point interior to $\triangle ABC$ (not on its sides), and lines AD, BD, CD meet BC, CA, AB at points P, Q, R respectively. Let M be the midpoint of PQ. Prove that if $\angle BRP = \angle PRC$, then $MR = MC$.

4. There are 900 students in an international school, including 59 international boys and 59 international girls. The students are partitioned into 30 classrooms, each having 30 students numbered from 1 to 30 so that either

   - any two international students in the same classroom do not have consecutive numbers, or
   - in all 30 classrooms, the student numbered 1 is a boy.

   Prove that there are four international students A, B, C, D of the same gender such that A, B come from the same classroom, C, D come from the same classroom (but different from A, B), and the difference between A and B's numbers is equal to the difference between C and D's numbers.

5. Find all positive integers n such that $n = d(n)^4$, where $d(n)$ denotes the number of positive divisors of n.

6. Let $x_1, x_2, \ldots, x_{2011}$ be real numbers in $[0, 1]$ and let $m = \frac{1}{2011}(x_1 + x_2 + \cdots + x_{2011})$ be their arithmetic mean. What is the maximum possible value of $\sum_{i=1}^{2011}(x_i - m)^2$?

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Day 2
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7. Let a, b, c, d be positive reals, and suppose that all roots of the equation $x^5 - ax^4 + bx^3 - cx^2 + dx = 1$ are real. Prove that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leqslant \frac{3}{5}$.

8. Let G be the centroid of a triangle $\triangle ABC$, and suppose that the line AC is tangent to the circumcircle of $\triangle ABG$. Prove that $AB + BC \leqslant 2AC$.

9. Prove that $\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$ is never an integer when $n > 1$ is a positive integer.

10. Does there exist a function f on the set of positive integers satisfying

$$f\big(m + f(n)\big) = f(m) + f(n) + f(n+1)$$

for all positive integers m, n?

11. In triangle $\triangle ABC$, the incircle is tangent to the sides BC, CA and AB at X, Y and Z respectively. Let I_a, I_b and I_c be the excenters of $\triangle ABC$ opposite A, B and C respectively. Prove that the incenter of $\triangle ABC$, the centroid of $\triangle I_aI_bI_c$, and the orthocenter of $\triangle XYZ$ lie on the same line.

12. 7662 chairs are placed in a circle around the city of Chiang Mai; they are also marked with a label for either 1st, 2nd, or 3rd grade students, so that there are 2554 chairs labeled with each label. The following situations happen, in order:

    (i) 2554 students each from the 1st, 2nd, and 3rd grades are given a ball as follows: 1st grade students receive footballs, 2nd grade students receive basketballs, and 3rd grade students receive volleyballs.

    (ii) The students go sit in chairs labeled for their grade.

    (iii) The students simultaneously send their balls to the student to their left, and this happens some positive number of times.

    A labeling of the chairs is called *lin-ping* if it is possible for all 1st, 2nd, and 3rd grade students to now hold volleyballs, footballs, and basketballs respectively. Compute the number of *lin-ping* labelings.