7th Thailand Mathematical Olympiad
Naresuan University, Phitsanulok
27 April 2010

Day 1
Time: 3 hours

Each problem is worth 2 points

1. Find the number of ways to distribute 11 balls into 5 boxes with different sizes, so that each box receives at least one ball, and the total number of balls in the largest and smallest boxes is more than the total number of balls in the remaining boxes.

2. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. A circle passing through B and C intersects sides AB and AC at D and E respectively. A point F on this circle is chosen so that $EF \perp BC$. If $BC = x, CF = y$, and $BF = z$, find the length of DF in terms of x, y, z.

3. Show that there are infinitely many positive integers n such that $2\underbrace{555\cdots55}_{n}3$ is divisible by 2553.

4. Let $\triangle ABC$ be an equilateral triangle, and let M and N be points on AB and AC, respectively, so that $AN = BM$ and $3MB = AB$. Lines CM and BN intersect at O. Find $\angle AOB$.

5. In a round-robin table tennis tournament between 2010 athletes, where each match ends with a winner and a loser, let $a_1, \ldots, a_{2010}$ denote the number of wins of each athlete, and let $b_1, \ldots, b_{2010}$ denote the number of losses of each athlete. Show that $a_1^2+a_2^2+\cdots+a_{2010}^2 = b_1^2 + b_2^2 + \cdots + b_{2010}^2$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the functional equation
$$f(3x + y) + f(3x - y) = f(x + y) + f(x - y) + 16f(x)$$
for all reals x, y. Show that f is even, that is, $f(-x) = f(x)$ for all reals x.

7. Let a, b, c be positive reals. Show that $\frac{a^5}{bc^2} + \frac{b^5}{ca^2} + \frac{c^5}{ab^2} \geqslant a^2 + b^2 + c^2$.

8. Define the modulo 2553 distance $d(x, y)$ between two integers x, y to be the smallest nonnegative integer d equivalent to either $x - y$ or $y - x$ modulo 2553. Show that, given a set S of integers such that $|S| \geqslant 70$, there must be $m, n \in S$ with $d(m, n) \leqslant 36$.

9. Let a, b, c be real numbers so that all roots of the equation
$$2x^5 + 5x^4 + 5x^3 + ax^2 + bx + c = 0$$
are real. Find the smallest real root of the equation above.

10. Find all primes p such that $\binom{100}{p} + 7$ is divisible by p.

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Day 2
Time: 4 hours

Each problem is worth 7 points

1. Show that, for every positive integer x, there is a positive integer $y \in \{2, 5, 13\}$ such that $xy - 1$ is not a perfect square.

2. The Ministry of Education selects 2010 students from 5 regions of Thailand to participate in a debate tournament, where each pair of students will debate in one of the three topics: politics, economics, and societal problems. Show that there are 3 students who were born in the same month, come from the same region, are of the same gender[1], and whose pairwise debates are on the same topic.

3. Let $\triangle ABC$ be a scalene triangle with $AB < BC < CA$. Let D be the projection of A onto the angle bisector of $\angle ABC$, and let E be the projection of A onto the angle bisector of $\angle ACB$. The line DE cuts sides AB and AC at M and N, respectively. Prove that $\frac{AB+AC}{BC} = \frac{DE}{MN} + 1$.

4. For $i = 1, 2$ let $\triangle A_iB_iC_i$ be a triangle with side lengths a_i, b_i, c_i and altitude lengths p_i, q_i, r_i. Define
$$a_3 = \sqrt{a_1^2 + a_2^2}, \quad b_3 = \sqrt{b_1^2 + b_2^2}, \quad \text{and} \quad c_3 = \sqrt{c_1^2 + c_2^2}.$$
Prove that a_3, b_3, c_3 are side lengths of a triangle, and if p_3, q_3, r_3 are the lengths of altitudes of this triangle, then
$$p_3^2 \geqslant p_1^2 + p_2^2, \quad q_3^2 \geqslant q_1^2 + q_2^2, \quad \text{and} \quad r_3^2 \geqslant r_1^2 + r_2^2.$$

5. Determine all functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the equation
$$f(x - t, y) + f(x + t, y) + f(x, y - t) + f(x, y + t) = 2010$$
for all real numbers x, y and for all nonzero t.

6. Show that no triples of primes p, q, r satisfy $p > r$, $q > r$, and $pq \mid r^p + r^q$.

[1] For the sake of this problem, assume there are two genders.