6th Thailand Mathematical Olympiad
Mahidol Wittayanusorn School, Nakhon Pathom
27 April 2009

Day 1
Time: 3 hours

Each problem is worth 2 points

1. Let $S \subset \mathbb{Z}^+$ be a set of positive integers with the following property: for any $a, b \in S$, if $a \neq b$ then $a + b$ is a perfect square. Given that $2009 \in S$ and $2087 \in S$, what is the maximum number of elements in S?

2. Let k and n be positive integers with $k < n$. Find the number of subsets of $\{1, 2, \ldots, n\}$ such that the difference between the largest and smallest elements in the subset is k.

3. Teeradet is a student in a class with 19 people. He and his classmates form clubs, so that each club must have at least one student, and each student can be in more than one club. Suppose that any two clubs differ by at least one student, and all clubs Teeradet is in have an odd number of students. What is the maximum possible number of clubs?

4. In triangle $\triangle ABC$, D is the midpoint of BC. Points E and F are chosen on side AC so that $AF = FE = EC$. Let AD intersect BE and BF and G and H, respectively. Find the ratio of the areas of $\triangle BGH$ and $\triangle ABC$.

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying:
$$f(xy + 2x + 2y - 1) = f(x)f(y) + f(y) + x - 2$$
for all real numbers x, y.

6. Let $\triangle ABC$ be a triangle with $AB > AC$; its incircle is tangent to BC at D. Let DE be a diameter of the incircle, and let F be the intersection between line AE and side BC. Find the ratio between the areas of $\triangle DEF$ and $\triangle ABC$ in terms of the three side lengths of $\triangle ABC$.

7. Let a, b, c be real numbers, and define $S_n = a^n + b^n + c^n$ for positive integers n. Suppose that S_1, S_2, S_3 are integers satisfying $6 \mid 5S_1 - 3S_2 - 2S_3$. Show that S_n is an integer for all positive integers n.

8. Let a, b, c be side lengths of a triangle, and define $s = \frac{a+b+c}{2}$. Prove that
$$\frac{2a(2a-s)}{b+c} + \frac{2b(2b-s)}{c+a} + \frac{2c(2c-s)}{a+b} \geqslant s.$$

9. In triangle $\triangle ABC$, D and E are midpoints of the sides BC and AC, respectively. Lines AD and BE are drawn intersecting at P. It turns out that $\angle CAD = 15^\circ$ and $\angle APB = 60^\circ$. What is the value of $\frac{AB}{BC}$?

10. Let $p \geqslant 5$ be a prime. Suppose that
$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots + \frac{1}{(p-1)^2} = \frac{a}{b}$$
where a/b is a fraction in lowest terms. Show that $p \mid a$.

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Day 2
Time: 4 hours

Each problem is worth 7 points

1. Let a and b be integers and p a prime. For each positive integer k, define $A_k = \{n \in \mathbb{Z}^+ \mid p^k \text{ divides } a^n - b^n\}$. Show that if A_1 is nonempty then A_k is nonempty for all positive integers k.

2. Is there an injective function $f : \mathbb{Z}^+ \to \mathbb{Q}$ satisfying the equation $f(xy) = f(x) + f(y)$ for all positive integers x and y?

3. Let $\square ABCD$ be a convex quadrilateral with the property that $MA \cdot MC + MA \cdot CD = MB \cdot MD$, where M is the intersection of the diagonals AC and BD. The angle bisector of $\angle ACD$ is drawn intersecting ray $\overrightarrow{BA}$ at K. Prove that $BC = DK$ if and only if $AB \parallel CD$.

4. Let k be a positive integer. Show that there are infinitely many positive integer solutions (m, n) to
$$(m-n)^2 = kmn + m + n.$$

5. A class contains 80 boys and 80 girls. On each weekday (Monday to Friday) of the week before final exams, the teacher has 16 books for the students to borrow, where a book can only be borrowed for one day at a time, and each student can only borrow once during the entire week. Show that there are two days and two books such that one of the following two statements is true:
   - (i) Both books were not borrowed on both days
   - (ii) Both books were borrowed on both days, and the four students who borrowed the books on these days are either all boys or all girls.

6. Find all polynomials of the form
$$P(x) = (-1)^n x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$$
with the following two properties:
   - (i) $\{a_1, a_2, \ldots, a_{n-1}, a_n\} = \{0, 1\}$, and
   - (ii) all roots of $P(x)$ are distinct real numbers.