5th Thailand Mathematical Olympiad[1]
Suankularb Wittayalai School, Bangkok
5 May 2008

Day 1
Time: 3 hours

Each problem is worth 2 points

1. Let $\triangle ABC$ be a triangle with $\angle BAC = 90^\circ$ and $\angle ABC = 60^\circ$. Point E is chosen on side BC so that $BE : EC = 3 : 2$. Compute $\cos \angle CAE$.

2. Let AD be the common chord of two equal-sized circles O_1 and O_2. Let B and C be points on O_1 and O_2, respectively, so that D lies on the segment BC. Assume that $AB = 15, AD = 13$ and $BC = 18$, what is the ratio between the inradii of $\triangle ABD$ and $\triangle ACD$?

3. Find all positive real solutions to the equation
$$x + \left\lfloor \frac{x}{3} \right\rfloor = \left\lfloor \frac{2x}{3} \right\rfloor + \left\lfloor \frac{3x}{5} \right\rfloor.$$

4. Prove that
$$\sqrt{a^2 + b^2 - \sqrt{2}ab} + \sqrt{b^2 + c^2 - \sqrt{2}bc} \geqslant \sqrt{a^2 + c^2}$$
for all real numbers $a, b, c > 0$.

5. Let $P(x)$ be a polynomial of degree 2008 with the following property: all roots of P are real, and for all real α, if $P(\alpha) = 0$ then $P(\alpha + 1) = 1$. Prove that P must have a repeated root.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the inequality $|f(x + y) - f(x) - f(y)| < 1$ for all reals x, y. Show that $\left| f\left(\frac{x}{2008}\right) - \frac{f(x)}{2008} \right| < 1$ for all real numbers x.

7. Two positive integers m, n satisfy the two equations
$$m^2 + n^2 = 3789 \quad \text{and} \quad \gcd(m, n) + \operatorname{lcm}(m, n) = 633.$$
Compute $m + n$.

8. Prove that $2551 \cdot 543^n - 2008 \cdot 7^n$ is never a perfect square, where n varies over the set of positive integers.

9. Find the number of pairs of sets (A, B) satisfying $A \subseteq B \subseteq \{1, 2, \ldots, 10\}$.

10. On the sides of triangle $\triangle ABC$, 17 points are added, so that there are 20 points in total (including the vertices of $\triangle ABC$.) What is the maximum possible number of (non-degenerate) triangles that can be formed by these points?

[1] The contest was officially held under the name *5th POSN Mathematical Olympiad.*

5th Thailand Mathematical Olympiad[2]
Suankularb Wittayalai School, Bangkok
6 May 2008

Day 2
Time: 4 hours

Each problem is worth 7 points

1. Let P be a point outside a circle ω. The tangents from P to ω are drawn touching ω at points A and B. Let M and N be the midpoints of AP and AB, respectively. Line MN is extended to cut ω at C so that N lies between M and C. Line PC intersect ω again at D, and lines ND and PB intersect at O. Prove that $\square MNOP$ is a rhombus.

2. Find all positive integers N with the following properties:

   (i) N has at least two distinct prime factors, and

   (ii) if $d_1 < d_2 < d_3 < d_4$ are the four smallest divisors of N then $N = d_1^2 + d_2^2 + d_3^2 + d_4^2$.

3. For each positive integer n, define $a_n = n(n+1)$. Prove that

$$n^{1/a_1} + n^{1/a_3} + n^{1/a_5} + \cdots + n^{1/a_{2n-1}} \geqslant n^{a_{3n+2}/a_{3n+1}}.$$

4. Let n be a positive integer. Show that

$$\binom{2n+1}{1} - \binom{2n+1}{3}2008 + \binom{2n+1}{5}2008^2 - \cdots + (-1)^n\binom{2n+1}{2n+1}2008^n$$

   is not divisible by 19.

5. Students in a class consisting of m boys and n girls line up. Over all possible ways of lining up, compute the average number of pairs of two boys or two girls who are next to each other.

6. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy $f(xy)^2 = f(x^2)f(y^2)$ for all positive reals x, y with $x^2y^3 > 2008$. Prove that $f(xy)^2 = f(x^2)f(y^2)$ for all positive reals x, y.

[2] The contest was officially held under the name *5th POSN Mathematical Olympiad.*