4th Thailand Mathematical Olympiad[1]
AFAPS, Nakhon Nayok
5 May 2007

Day 1: Short Answer
Time: 3 hours

Each problem is worth 1 point

1. In a circle $\odot O$, radius OA is perpendicular to radius OB. Chord AC intersects OB at E so that the length of arc AC is one-third the circumference of $\odot O$. Point D is chosen on OB so that $CD \perp AB$. Suppose that segment AC is 2 units longer than segment OD. What is the length of segment AC?

2. Let $\square ABCD$ be a cyclic quadrilateral so that arcs AB and BC are equal. Given that $AD = 6$, $BD = 4$ and $CD = 1$, compute AB.

3. A triangle $\triangle ABC$ has $\angle B = 90^\circ$. A circle is tangent to AB at B and also tangent to AC. Another circle is tangent to the first circle as well as the two sides AB and AC. Suppose that $AB = \sqrt{3}$ and $BC = 3$. What is the radius of the second circle?

4. A triangle $\triangle ABC$ has $AC = 16$ and $BC = 12$. E and F are points on AC and BC, respectively, so that $CE = 3CF$. Let M be the midpoint of AB, and let lines EF and CM intersect at G. Compute the ratio $EG : GF$.

5. A triangle $\triangle ABC$ has $\angle A = 90^\circ$, and a point D is chosen on AC. Point F is the foot of altitude from A to BC. Suppose that $BD = DC = CF = 2$. Compute AC.

6. Let M be the midpoint of a given segment BC. Point A is chosen to maximize $\angle ABC$ while subject to the condition that $\angle MAC = 20^\circ$. What is the ratio BC/BA?

7. Let a, b, c be complex numbers such that $a+b+c = 1$, $a^2+b^2+c^2 = 2$ and $a^3+b^3+c^3 = 3$. Find the value of $a^4 + b^4 + c^4$.

8. Let $x_1, x_2, \ldots, x_{84}$ be the roots of the equation $x^{84} + 7x - 6 = 0$. Compute $\sum_{k=1}^{84} \frac{x_k}{x_k - 1}$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the equation $f(x^2 + x + 3) + 2f(x^2 - 3x + 5) = 6x^2 - 10x + 17$ for all real numbers x. What is the value of $f(85)$?

10. Find the smallest positive integer n such that the equation $\sqrt{3}z^{n+1} - z^n - 1 = 0$ has a root on the unit circle.

11. Compute the number of functions $f : \{1, 2, \ldots, 2550\} \to \{61, 80, 84\}$ such that $\sum_{k=1}^{2550} f(k)$ is divisible by 3.

12. An alien with four feet wants to wear four identical socks and four identical shoes, where on each foot a sock must be put on before a shoe. How many ways are there for the alien to wear socks and shoes?

13. Let $S = \{1, 2, \ldots, 8\}$. How many ways are there to select two disjoint subsets of S?

14. The sum
$$\sum_{k=84}^{8000} \binom{k}{84} \binom{8084-k}{84}$$
can be written as a binomial coefficient $\binom{a}{b}$ for integers a, b. Find a possible pair (a, b).

[1] The contest was officially held under the name *4th POSN Mathematical Olympiad.*

15. Compute the remainder when $222!^{111} + 111^{222!} + 111!^{222} + 222^{111!}$ is divided by 2007.
16. What is the smallest positive integer with 24 positive divisors?
17. Compute the product of positive integers n such that $n^2 + 59n + 881$ is a perfect square.
18. Let p_k be the k^{th} prime number. Find the remainder when $\sum_{k=2}^{2550} p_k^{p_k^4-1}$ is divided by 2550.

4th Thailand Mathematical Olympiad[2]
AFAPS, Nakhon Nayok
6 May 2008

Day 2: Proof-based
Time: 4 hours

Each problem is worth 7 points

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the inequality
$$\sum_{i=1}^{2549} f(x_i + x_{i+1}) + f\left(\sum_{i=1}^{2550} x_i\right) \leqslant \sum_{i=1}^{2550} f(2x_i)$$
for all reals $x_1, x_2, \ldots, x_{2550}$.

2. In a dance party there are n girls and n boys, and some m songs are played. Each song is danced to by at least one pair of a boy and a girl, who both receive a *malai* each. Prove that for all positive integers $k \leqslant n$, it is possible to select k boys and $n-k$ girls so that the n selected people received at least m malai in total.

3. Two circles intersect at X and Y. The line through the centers of the circles intersect the first circle at A and C, and intersect the second circle at B and D so that A, B, C, D lie in this order. The common chord XY cuts BC at P, and a point O is arbitrarily chosen on segment XP. Lines CO and BO are extended to intersect the first and second circles at M and N, respectively. If lines AM and DN intersect at Z, prove that X, Y and Z lie on the same line.

4. Find all primes p such that $\frac{2^{p-1}-1}{p}$ is a perfect square

5. The freshman class of a school consists of 229 boys and 271 girls, and is divided into 10 rooms of 50 students each; the students in each room are numbered from 1 to 50. The physical education teacher wants to select a relay running team consisting of 1 boy and 3 girls or 1 girl and 3 boys, so that the four students must be two pairs of students with the same number from two rooms. Show that the number of possible teams is odd.

6. A triangle has perimeter $2s$, inradius r, and incenter I. If s_a, s_b and s_c are the distances from I to the three vertices, then show that
$$\frac{3}{4} + \frac{r}{s_a} + \frac{r}{s_b} + \frac{r}{s_c} \leqslant \frac{s^2}{12r^2}.$$

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