3rd Thailand Mathematical Olympiad[1]
Naresuan University, Phitsanulok
9 May 2006

Day 1: Short Answer
Time: 3 hours

Each problem is worth 1 point

1. Let O be the circumcenter of a triangle $\triangle ABC$. It is given that $\angle ABC = 70°, \angle ACB = 50°$. Let the angle bisector of $\angle BAC$ intersect the circumcircle of $\triangle ABC$ again at D. Compute $\angle ADO$.

2. Triangle $\triangle ABC$ has side lengths $AB = 2, CA = 3$ and $BC = 4$. Compute the radius of the circle centered on BC that is tangent to both AB and AC.

3. The three medians of a triangle has lengths $3, 4, 5$. What is the length of the shortest side of this triangle?

4. Let P be a point outside a circle centered at O. From P, tangent lines are drawn to the circle, touching the circle at points A and B. Ray $\overrightarrow{BO}$ is drawn intersecting the circle again at C and intersecting ray $\overrightarrow{PA}$ at Q. If $3QA = 2AP$, what is the value of $\sin \angle CAQ$?

5. Let $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ satisfy the functional equation
$$f(m^2 + n^2) = \big(f(m) - f(n)\big)^2 + f(2mn)$$
for all nonnegative integers m, n. If $8f(0) + 9f(1) = 2006$, compute $f(0)$.

6. A function $f : \mathbb{R} \to \mathbb{R}$ has $f(1) < 0$, and satisfy the functional equation
$$f\big(\cos(x + y)\big) = (\cos x)f(\cos y) + 2f(\sin x)f(\sin y)$$
for all reals x, y. Compute $f\left(\frac{2006}{2549}\right)$.

7. Let x, y, z be reals summing to 1 which minimizes $2x^2 + 3y^2 + 4z^2$. Find x.

8. Let a, b, c be the roots of the equation $x^3 - 9x^2 + 11x - 1 = 0$, and define $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$. Compute $s^4 - 18s^2 - 8s$.

9. Compute the largest integer not exceeding $\frac{2549^3}{2547 \cdot 2548} - \frac{2547^3}{2548 \cdot 2549}$.

10. Find the remainder when $26!^{26} + 27!^{27}$ is divided by 29.

11. Let p_n be the n^{th} prime number. Find the remainder when $\prod_{n=1}^{2549} 2006^{p_n^2 - 1}$ is divided by 13.

12. Let $a_n = 2^{3n-1} + 3^{6n-2} + 5^{6n-3}$. Compute $\gcd(a_1, a_2, \ldots, a_{2549})$.

13. Compute the remainder when $\underbrace{111\ldots1}_{1862}$ is divided by 2006.

14. Find the smallest positive integer n such that $2549 \mid n^{2545} - 2541$.

15. How many positive integers $n < 2549$ are there such that $x^2 + x - n$ has an integer root?

16. Find the number of triples of sets (A, B, C) such that $A \cup B \cup C = \{1, 2, 3, \ldots, 2549\}$.

[1] The contest was officially held under the name *3rd POSN Mathematical Olympiad.*

17. Six people, with distinct weights, want to form a triangular position where there are three people in the bottom row, two in the middle row, and one in the top row, and each person in the top two rows must weigh less than both of their supports. How many distinct formations are there?

18. In May, the traffic police wants to select 10 days to patrol, but no two consecutive days can be selected. How many ways are there for the traffic police to select patrol days?

3rd Thailand Mathematical Olympiad[2]
Naresuan University, Phitsanulok
10 May 2006

Day 2: Proof-based
Time: 4 hours

Each problem is worth 7 points

1. Show that the product of three consecutive positive integers is never a perfect square.

2. From a point P outside a circle, two tangents are drawn touching the circle at points A and C. Let B be a point on segment AC, and let segment PB intersect the circle at point Q. The angle bisector of $\angle AQC$ intersects segment AC at R. Show that $\frac{AB}{BC} = \left(\frac{AR}{RC}\right)^2$.

3. Let $P(x), Q(x)$ and $R(x)$ be polynomials satisfying the equation
$$2xP(x^3) + Q(-x - x^3) = (1 + x + x^2)R(x).$$
Show that $x - 1$ divides $P(x) - Q(x)$.

4. In a classroom, 28 students are divided into 4 groups of 7, and in each group the students are labeled $1, 2, \ldots, 7$ in some order. Show that no matter how the labels are assigned, there must be four students of the same gender[3] who come from two groups and share the same two labels.

5. Show that there are coprime positive integers m and n such that
$$2549 \mid (25 \cdot 49)^m + 25^n - 2 \cdot 49^n.$$

6. Let a, b, c be positive reals. Show that
$$1 + \frac{3}{ab + bc + ca} \geqslant \frac{6}{a + b + c}.$$

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[3] For the sake of this problem, assume there are two genders.