2nd Thailand Mathematical Olympiad[1]
Ubon Ratchathani University, Ubon Ratchathani
3 May 2005

Day 1: Short Answer
Time: 3 hours

Each problem is worth 1 point

1. Let $\square ABCD$ be a trapezoid inscribed in a unit circle with diameter AB. If $DC = 4AD$, compute AD.

2. Let $\triangle ABC$ be an acute triangle, and let A' and B' be the feet of altitudes from A to BC and from B to CA, respectively; the altitudes intersect at H. If BH is equal to the circumradius of $\triangle ABC$, find $\frac{A'B}{AB}$.

3. Triangle $\triangle ABC$ is isosceles with $AB = AC$ and $\angle ABC = 2\angle BAC$. Compute $\frac{AB}{BC}$.

4. Triangle $\triangle ABC$ is inscribed in the circle with diameter BC. If $AB = 3, AC = 4$, and O is the *incenter* of $\triangle ABC$, then find $BO \cdot OC$.

5. A die is thrown six times. How many ways are there for the six rolls to sum to 21?

6. Find the number of positive integer solutions to the equation $(x_1+x_2+x_3)^2(y_1+y_2) = 2548$.

7. How many ways are there to express 2548 as a sum of at least two positive integers, where two sums that differ in order are considered different?

8. For each subset T of $S = \{1, 2, \dots, 7\}$, the *result* $r(T)$ of T is computed as follows: the elements of T are written, largest to smallest, and alternating signs $(+, -)$ starting with $+$ are put in front of each number. The value of the resulting expression is $r(T)$. (For example, for $T = \{2, 4, 7\}$, we have $r(T) = +7 - 4 + 2 = 5$.) Compute the sum of $r(T)$ as T ranges over all subsets of S.

9. Compute $\gcd\left(\frac{135^{90}-45^{90}}{90^2}, 90^2\right)$.

10. What is the remainder when $\sum_{k=1}^{2005} k^{2005\cdot 2^{2005}}$ is divided by 2^{2005}?

11. Find the smallest positive integer x such that 2^{2548} divides $x^{2005} + 1$.

12. Find the number of even integers n such that $0 \leqslant n \leqslant 100$ and $5 \mid n^2 \cdot 2^{2n^2} + 1$.

13. Find all odd integers k for which there exists a positive integer m satisfying the equation
$$k + (k+5) + (k+10) + \cdots + \big(k + 5(m-1)\big) = 1372.$$

14. A function $f : \mathbb{Z}^+ \to \mathbb{Z}$ is given so that
$$f(m+n) = f(m) + f(n) + 2mn - 2548$$
for all positive integers m, n. Given that $f(2548) = -2548$, find the value of $f(2)$.

15. A function $f : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation
$$f(x+2y) + 2f(y-2x) = 3x - 4y + 6$$
for all reals x, y. Compute $f(2548)$.

[1] The contest was officially held under the name *2nd POSN Mathematical Olympiad.*

16. Compute the sum of roots of $(2-x)^{2005} + x^{2005} = 0$.

17. For $a, b \geqslant 0$ we define $a * b = \frac{a+b+1}{ab+12}$. Compute $0 * (1 * (2 * (\cdots (2003 * (2004 * 2005)) \cdots)))$.

18. Compute the sum
$$\sum_{k=0}^{1273} \frac{1}{1 + \tan^{2548}\left(\frac{k\pi}{2548}\right)}.$$

19. Let $P(x)$ be a monic polynomial of degree 4 such that for $k = 1, 2, 3$, the remainder when $P(x)$ is divided by $x - k$ is equal to k. Find the value of $P(4) + P(0)$.

20. Let $a, b, c, d > 0$ satisfy $36a + 4b + 4c + 3d = 25$. What is the maximum possible value of $ab^{1/2}c^{1/3}d^{1/4}$?

21. Compute the minimum value of
$$\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha)$$
as α, β, γ ranges over the real numbers.

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Day 2: Proof-based
Time: 4 hours

Each problem is worth 7 points

1. A point A is chosen outside a circle with diameter BC so that $\triangle ABC$ is acute. Segments AB and AC intersect the circle at D and E, respectively, and CD intersects BE at F. Line AF intersects the circle again at G and intersects BC at H. Prove that $AH \cdot FH = GH^2$.

2. Let S be a set of three distinct integers. Show that there are $a, b \in S$ such that $a \neq b$ and $10 \mid a^3b - ab^3$.

3. Does there exist a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(f(n)) = 2n$ for all positive integers n? Justify your answer, and if the answer is *yes*, give an explicit construction.

4. Let O_1 be the center of a semicircle ω_1 with diameter AB and let O_2 be the center of a circle ω_2 inscribed in ω_1 and which is tangent to AB at O_1. Let O_3 be a point on AB that is the center of a semicircle ω_3 which is tangent to both ω_1 and ω_2. Let P be the intersection of the line through O_3 perpendicular to AB and the line through O_2 parallel to AB. Show that P is the center of a circle Γ tangent to all of ω_1, ω_2 and ω_3.

5. During the morning of the first day of class, 50 new students line up, and each student make friends with the students next to them; no other pairs of friendships are made. That same day, in the afternoon, a teacher wants the 50 students to line up again so that for every student S (except the first in line), at least one friend of S is in front of S. How many ways are there for the students to line up in the afternoon?

6. Let a, b, c be distinct real numbers. Prove that

$$\left(\frac{2a-b}{a-b}\right)^2 + \left(\frac{2b-c}{b-c}\right)^2 + \left(\frac{2c-a}{c-a}\right)^2 \geqslant 5.$$

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