Thailand October Camp 2015 IPST, Bangkok 22 October 2015

Algebra and Number Theory Exam Time: 4.5 hours

1. Find all polynomials $P \in \mathbb{Z}[x]$ such that

$$|P(x) - x| \leqslant x^2 + 1$$

for all real numbers x.

2. Determine all positive integers M such that the sequence a_0, a_1, a_2, \cdots defined by

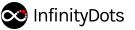
$$a_0 = M + \frac{1}{2}$$
 and $a_{k+1} = a_k \lfloor a_k \rfloor$ for $k = 0, 1, 2, \cdots$

contains at least one integer term.

3. Determine whether there exists a positive integer a such that

$$2015a, 2016a, \ldots, 2558a$$

are all perfect powers.



Inequalities and Combinatorics Exam Time: 4.5 hours

- 1. (4 points) Let a_1, a_2, a_3, \ldots be a sequence of positive integers such that
 - (i) $a_1 = 0$

(ii) for all
$$i \ge 1$$
, $a_{i+1} = a_i + 1$ or $-a_i - 1$.

Prove that $\frac{a_1+a_2+\cdots+a_n}{n} \ge -\frac{1}{2}$ for all $n \ge 1$.

- 2. (3 points) Find the number of sequences $a_1, a_2, \ldots, a_{100}$ such that
 - (i) There exists $i \in \{1, 2, ..., 100\}$ such that $a_i = 3$, and
 - (ii) $|a_i a_{i+1}| \leq 1$ for all $1 \leq i < 100$.
- 3. (7 points) Find all positive integers $n \ge 3$ such that it is possible to triangulate a convex *n*-gon such that all vertices of the *n*-gon have even degree.
- 4. (7 points) Let a, b, c be positive reals such that $4(a + b + c) \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Define

$$A = \sqrt{\frac{3a}{a + 2\sqrt{bc}}} + \sqrt{\frac{3b}{b + 2\sqrt{ca}}} + \sqrt{\frac{3c}{c + 2\sqrt{ab}}}$$
$$B = \sqrt{a} + \sqrt{b} + \sqrt{c}$$
$$C = \frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}}.$$

Prove that

$$A \leqslant 2B \leqslant 4C.$$



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Functional Equations and Geometry Exam Time: 4.5 hours

1. Find all functions $f: \mathbb{Q} \to \mathbb{Q}$ such that

$$f(xy) + f(x+y) = f(x)f(y) + f(x) + f(y)$$

for all $x, y \in \mathbb{Q}$.

- 2. Let ω be a circle touching two parallel lines ℓ_1, ℓ_2, ω_1 a circle touching ℓ_1 at A and ω externally at C, and ω_2 a circle touching ℓ_2 at B, ω externally at D, and ω_1 externally at E. Prove that AD, BC intersect at the circumcenter of $\triangle CDE$.
- 3. Let *H* be the orthocenter of acute-angled $\triangle ABC$, and *X*, *Y* points on the ray *AB*, *AC*. (*B* lies between *X*, *A*, and *C* lies between *Y*, *A*.) Lines *HX*, *HY* intersect *BC* at *D*, *E* respectively. Let the line through *D* parallel to *AC* intersect *XY* at *Z*. Prove that $\angle XHY = 90^{\circ}$ if and only if *ZE* || *AB*.

