Thailand October Camp 2015
IPST, Bangkok
22 October 2015

Algebra and Number Theory Exam
Time: 4.5 hours

1. Find all polynomials $P \in \mathbb{Z}[x]$ such that

$$|P(x) - x| \leqslant x^2 + 1$$

   for all real numbers x.

2. Determine all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

   contains at least one integer term.

3. Determine whether there exists a positive integer a such that

$$2015a, 2016a, \ldots, 2558a$$

   are all perfect powers.

Thailand October Camp 2015
IPST, Bangkok
28 October 2015

Inequalities and Combinatorics Exam
Time: 4.5 hours

1. *(4 points)* Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers such that

   (i) $a_1 = 0$

   (ii) for all $i \geqslant 1$, $a_{i+1} = a_i + 1$ or $-a_i - 1$.

   Prove that $\frac{a_1+a_2+\cdots+a_n}{n} \geqslant -\frac{1}{2}$ for all $n \geqslant 1$.

2. *(3 points)* Find the number of sequences $a_1, a_2, \ldots, a_{100}$ such that

   (i) There exists $i \in \{1, 2, \ldots, 100\}$ such that $a_i = 3$, and

   (ii) $|a_i - a_{i+1}| \leqslant 1$ for all $1 \leqslant i < 100$.

3. *(7 points)* Find all positive integers $n \geqslant 3$ such that it is possible to triangulate a convex n-gon such that all vertices of the n-gon have even degree.

4. *(7 points)* Let a, b, c be positive reals such that $4(a+b+c) \geqslant \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Define

$$\begin{aligned} A &= \sqrt{\frac{3a}{a+2\sqrt{bc}}} + \sqrt{\frac{3b}{b+2\sqrt{ca}}} + \sqrt{\frac{3c}{c+2\sqrt{ab}}} \\ B &= \sqrt{a} + \sqrt{b} + \sqrt{c} \\ C &= \frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}}. \end{aligned}$$

   Prove that

$$A \leqslant 2B \leqslant 4C.$$

Thailand October Camp 2015
IPST, Bangkok
30 October 2015

Functional Equations and Geometry Exam
Time: 4.5 hours

1. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(xy) + f(x + y) = f(x)f(y) + f(x) + f(y)$$

for all $x, y \in \mathbb{Q}$.

2. Let ω be a circle touching two parallel lines ℓ_1, ℓ_2, ω_1 a circle touching ℓ_1 at A and ω externally at C, and ω_2 a circle touching ℓ_2 at B, ω externally at D, and ω_1 externally at E. Prove that AD, BC intersect at the circumcenter of $\triangle CDE$.

3. Let H be the orthocenter of acute-angled $\triangle ABC$, and X, Y points on the ray AB, AC. (B lies between X, A, and C lies between Y, A.) Lines HX, HY intersect BC at D, E respectively. Let the line through D parallel to AC intersect XY at Z. Prove that $\angle XHY = 90°$ if and only if $ZE \parallel AB$.