2018 Thailand TST Solutions

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Contents

1 General Information

§1.1 (Important) Notes of Confidentiality

This handout contains many problems from IMO Shortlist 2017, which must be kept confidential until the end of IMO 2018. **Do not disclose this handout to anyone except the students who took the TST and instructors of Thailand TST Camp until the end of IMO 2018.**

§1.2 Thailand TST 2018 Information

Thailand TST is a set of exams held throughout the training camps in order to select six students out of 25 to make the IMO team. This year, there were nine exams held in the following dates.

- Day 1 : 25 December 2017
- Day 2 : 29 January 2018
- Day 3 : 30 January 2018
- Day 4 : 11 March 2018
- Day 5 : 19 March 2018
- Day 6 : 20 March 2018
- Day 7 : 27 April 2018
- Day 8 : 29 April 2018
- Day 9 : 30 April 2018

Each test had 3 problems to be solved in 4.5 hours time limit (more specifically from 9.00 am to 1.30 pm). Except the first test which had 4 problems instead of 3.

§1.3 Tests Comments

The subject distribution turns out to be the following. (Day 1 is not counted in the upper table.)

Subject Distribution by Problem Number

	D2	D3	D4	D5	D6	D7	D8	D9	Total
P1	N	C	A	G	G	A	C	N	AA CC GG NN
P2	C	G	C	N	N	C	A	G	A CCC GG NN
P3	A	N	G	C	A	N	G	A	AAA C GG NN

Subject Distribution by Difficulty

Subject	Easy	Med	Hard
Algebra	AAA	A	AAA
Combinatorics	CCC	CCC	C
Number Theory	NNN	NNN	N
Geometry	GGG	GG	GG

§1.4 Acknowledgement

Thanks to all Thailand TST Camp instructors, who make the team selection tests possible, and thanks to everyone to contribute solutions or help editing this handout.

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2 Problems

§2.1 Day 1 Problems

Problem 1. Let $x, y, z \in \mathbb{R}^+$ such that $xyz = 1$. Prove that

$$\sum_{cyc} \frac{1}{\sqrt{x+2y+6}} \leqslant \sum_{cyc} \frac{x}{\sqrt{x^2+4\sqrt{y}+4\sqrt{z}}}.$$

Problem 2. Let $n < 2017$ be a fixed positive integer. Exactly n of the vertices of a regular 2017-gon are coloured red, and the other vertices are coloured blue. Prove that the number of isosceles triangles which vertices are the same color only depend on n, but not the configuration of red and blue points.

Problem 3. Do there exists an arithmetic progression with 2017 terms which each term is not perfect p ower but the product of all 2017 terms is perfect power?

Problem 4. Let ΔABC be an acute triangle with altitudes AA_1, BB_1, CC_1 and orthocenter H. Let K, L be the midpoints of BC_1, CB_1. Let ℓ_A be the external angle bisector of $\angle BAC$. Let ℓ_B, ℓ_C be the lines through B, C and perpendicular to ℓ_A. Let ℓ_H be the line through H, parallel to ℓ_A. Prove that the centers of circumcircle of $\Delta A_1B_1C_1, \Delta AKL$ and the rectangle formed by $\ell_A, \ell_B, \ell_C, \ell_H$ are colinear.

§2.2 Day 2 Problems

Problem 1. Determine all positive integer n such that for any positive integers $a_1, a_2, ..., a_n$ which their sum is not divisible by n, there exists indices i such that each of

$$a_i,\ a_i + a_{i+1},\ a_i + a_{i+1} + a_{i+2}, ...,\ a_i + a_{i+1} + ... + a_{i+n-1}$$

is not divisible by n where $a_i = a_{i-n}$ for any $i > n$.

Problem 2. For any finite set $M \subset \mathbb{Z}^+$ and set $A \subseteq M$, define

$$f_M(A) = \{x \in M | x \text{ is divisible by odd number of elements of } A\}$$

For any positive integer k, we call M k-colorable if and only if every subsets of M can be assigned one of k colours so that for any $A \subseteq M$ such that $f_M(A) \neq A$, sets $f_M(A)$ and A must be different color.

Determine the least positive integer k such that every finite set $M \subset \mathbb{Z}^+$ is k-colorable.

Problem 3. Let S be a finite set and let $f : S \to S$ be a function. Suppose that for any function $g : S \to S$, functions $f \circ g \circ f$ and $g \circ f \circ g$ are different. Prove that any elements of S which is in the range of f must be in the range of $f \circ f$.

§2.3 Day 3 Problems

Problem 1. Rectangle $\mathcal{R}$ which its sides are positive odd integer is tiled by some small rectangles, each of them has integer sides. Prove that there is one of those small rectangles which the four distances from one of its sides to the closest side of $\mathcal{R}$ are all odd or even.

Problem 2. Let O, H be the circumcenter and orthocenter of triangle ABC. Select points P, Q on the line AO such that $BP \perp AC$ and $CQ \perp AB$. Prove that circumcenter of triangle PQH lies on one of the medians of triangle ABC.

Problem 3. Determine the smallest positive integer n such that there exists infinitely many n-tuple of positive rational numbers $(a_1, a_2, ..., a_n)$ such that

$$a_1 + a_2 + ... + a_n,\ \frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}$$

are integer.

§2.4 Day 4 Problems

Problem 1. Let $a_1, a_2, ..., a_n$ be positive integers, not all equal to 1. Let $M = a_1 a_2 ... a_n$ and $k = \frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}$. Suppose that k is integer. Prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive real roots.

Problem 2. Let $(x_1, x_2, ..., x_{100})$ be a permutation of $\{1, 2, ..., 100\}$. Define

$$S = \{m \mid m \text{ is the median of } \{x_i, x_{i+1}, x_{i+2}\} \text{ for some } i\}.$$

Determine the minimum possible value of the sum of all elements of S.

Problem 3. Let $ABCC_1B_1A_1$ be a convex hexagon such that $BA = BC$ and lines AA_1, BB_1 and CC_1 share the same perpendicular bisector. Let ω denotes circumcircle of ΔABC. Diagonals AC_1 and A_1C meet at D. Circumcircles of triangle A_1BC_1 intersects ω at $E \neq A$. Prove that lines BB_1 and DE intersects on ω.

§2.5 Day 5 Problems

Problem 1. Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$ and $\angle EDC = \angle ABC$. Prove that lines AC, BD and the line from E perpendicular to BC are concurrent.

Problem 2. Determine all ordered pairs (p, q) of prime numbers which $p > q$ and

$$\frac{(p+q)^{p+q}(p-q)^{p-q}-1}{(p+q)^{p-q}(p-q)^{p+q}-1}$$

is an integer.

Problem 3. Let n be a positive integer. Consider an $n \times n \times n$ cube, which each of n^3 cubes is colored. Define a *box* as an $n \times n \times 1$ grid of cubes, in any of three possible orientation. We also define the *color-set* of each box as the set of all color. Suppose that for any box $\mathcal{B}_1$, there exists another two boxes $\mathcal{B}_2, \mathcal{B}_3$ such that $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ have different orientations but having same color-set. Determine the maximum possible number of colors used.

§2.6 Day 6 Problems

Problem 1. Let ABC be a triangle with points E, F lies on segment BC. Let K, L be points on segments AB, AC respectively such that $EK \parallel AC$, $FL \parallel AB$. The incircle of triangles BEK, CFL touches segments AB, AC at X, Y respectively. Lines AC and EX intersect at M and lines AB and FY intersects at N. Suppose that $AX = AY$. Prove that $MN \parallel BC$.

Problem 2. Call a rational number *short* if it has terminating decimal. For a positive integer m, call a positive integer t *m-tastic* if and only if there exists $c \in \{1, 2, ..., 2017\}$ which $\frac{10^t-1}{cm}$ is short but $\frac{10^k-1}{cm}$ is not short for any positive integer $k < t$.

Let $S(m)$ be the number of all m-tastic integer. Determine the maximum possible value of $S(m)$ over all positive integer m.

Problem 3. Let $n \geqslant 3$ be an integer. Let $a_1, a_2, ..., a_n \in [0,1]$ which $a_1 + a_2 + \cdots + a_n = 2$. Prove that

$$\sqrt{1-\sqrt{a_1}} + \sqrt{1-\sqrt{a_2}} + \cdots + \sqrt{1-\sqrt{a_n}} \leqslant n - 3 + \sqrt{9 - 3\sqrt{6}}.$$

§2.7 Day 7 Problems

Problem 1. Determine all functions $g : \mathbb{R} \to \mathbb{R}$ which there exists strictly monotone function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x)g(y) + f(y)$$

for all reals x, y.

Problem 2. Sir Alex plays the following game in a row of 9 cells. At the beginning, all cells are empty. In each move, he performs exactly one of the following procedures.

i) He chooses one empty cells and inserts integer in form 2^k for some integer $k \geqslant 0$.

ii) He chooses two (not necessarily adjacent) cells which have the same value 2^k. He replaces one of the two cells with 2^{k+1} and erase the number in other cell.

Suppose that eventually, Sir Alex ends up with exactly one cells containing number 2^n for some positive integer n while all other cells are empty. Determine (in terms of n) the maximum possible number of moves he could have made.

Problem 3. Let n be a fixed odd positive integer. For an odd prime p, define

$$a_p = \frac{\sum_{k=1}^{\frac{p-1}{2}} \left\{ \frac{k^{2n}}{p} \right\}}{p-1}$$

Prove that a_p yields equal value for infinitely many primes p.

Notes : For any real number x, define $\{x\} = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the greatest integer not exceeding x

§2.8 Day 8 Problems

Problem 1. Let n be a fixed positive integer. Define a *chameleon* as any word of $3n$ letters which there are exactly n occurrences of letters a, b, c. Define a *swap* as a transposition of any two adjacent letters. Prove that for any chameleon X, there exists chameleon Y which cannot be reached from X using fewer than $3n^2/2$ swaps.

Problem 2. Let $a_1, a_2, a_3, ...$ be the sequence of real numbers satisfying

$$a_n = -\max_{i+j=n}(a_i + a_j) \quad \text{for any } n > 2017.$$

Prove that this sequence is bounded i.e., there exists real number M such that $|a_n| < M$ for any positive integer n.

Problem 3. Let $ABCD$ be a convex quadrilateral which has an inscribed circle centered at I. Let I_A, I_B, I_C, I_D be the incenters of triangles ABD, BCA, CBD, DAC respectively. The common external tangents of circumcircles of triangles AI_BI_D and CI_BI_D intersects at X. The common external tangents of circumcircles of triangles BI_AI_C and DI_AI_C intersects at Y. Prove that $\angle XIY = 90°$.

§2.9 Day 9 Problems

Problem 1. Let $p \geqslant 2$ be prime number. Alice and Bob play a game which both players alternately make moves where Alice goes first. In each turn, player choose index i from the set $\{0, 1, ..., p-1\}$ that no one choose before then choose a_i from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. When all of the numbers $a_0, a_1, ..., a_{p-1}$ have been chosen, the game is ended and the value

$$M = a_0 + a_1 \cdot 10 + \cdots + a_{p-1} \cdot 10^{p-1}$$

is computed. Alice's goal is to make M divisible by p while Bob's goal is to prevent this. Prove that Alice has a winning strategy.

Problem 2. Let ω be the A-excircle of ΔABC, which touches lines BC, CA, AB at D, E, F respectively. Circumcircle of triangle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that circumcircle of triangle MPQ is tangent to ω.

Problem 3. Let $n \geqslant 3$ be a positive integer. Call an n-tuple of real numbers $(x_1, x_2, ..., x_n)$ *shiny* if and only if for any permutation $(y_1, y_2, ..., y_n)$ of those number, we have

$$y_1y_2 + y_2y_3 + ... + y_{n-1}y_n \geqslant -1.$$

Determine the largest constant $K = K(n)$ such that the inequality

$$\sum_{1 \leqslant i < j \leqslant n} x_ix_j \geqslant K$$

holds for all shiny n-tuple $(x_1, x_2, ..., x_n)$.

3 Solutions

§3.1 Day 1 Solutions

Problem 1. Let $x, y, z \in \mathbb{R}^+$ such that $xyz = 1$. Prove that

$$\sum_{\text{cyc}} \frac{1}{\sqrt{x+2y+6}} \leqslant \sum_{\text{cyc}} \frac{x}{\sqrt{x^2+4\sqrt{y}+4\sqrt{z}}}.$$

Solution 1. (Jirayus Jinapong)

We claim that

$$\sum_{\text{cyc}} \frac{1}{\sqrt{x+2y+6}} \leqslant 1 \leqslant \sum_{\text{cyc}} \frac{x}{\sqrt{x^2+4\sqrt{y}+4\sqrt{z}}}$$

From Cauchy-Schwarz Inequality, we get

$$\sum_{\text{cyc}} \sqrt{\frac{1}{x+2y+6}}\sqrt{1} \leqslant \sqrt{3}\sqrt{\sum_{\text{cyc}} \frac{1}{x+2y+6}}$$

Next, by AM-HM Inequality, we get

$$\sum_{\text{cyc}} \frac{1}{(x+2)+(y+2)+(y+2)} \leqslant \sum_{\text{cyc}} \frac{\frac{1}{x+2}+\frac{2}{y+2}}{9}$$

$$\sum_{\text{cyc}} \frac{1}{x+2y+6} \leqslant \frac{\sum\limits_{\text{cyc}} \frac{1}{x+2}}{3}$$

From

$$\sum_{\text{cyc}} \frac{1}{x+2} = \frac{\sum\limits_{\text{cyc}} xy + 4\sum\limits_{\text{cyc}} x + 12}{9 + 2\sum\limits_{\text{cyc}} xy + 4\sum\limits_{\text{cyc}} x}$$

and

$$\sum_{\text{cyc}} xy \geqslant 3\sqrt[3]{x^2y^2z^2} = 3$$

Therefore,

$$\sum_{\text{cyc}} \frac{1}{x+2y+6} \leqslant 1$$

On the other hand, From

$$\begin{aligned}
\sqrt{x^2+4\sqrt{y}+4\sqrt{z}} &= \sqrt{x^2+4\sqrt[4]{(y^2)(xy)(xz)(yz)}+4\sqrt[4]{(z^2)(xy)(xz)(yz)}} \\
&\geqslant \sqrt{x^2+(y^2+xy+xz+yz)+(z^2+xy+xz+yz)} \\
&\geqslant x+y+z
\end{aligned}$$

Hence,

$$\sum_{\text{cyc}} \frac{x}{\sqrt{x^2+4\sqrt{y}+4\sqrt{z}}} \geqslant \sum_{\text{cyc}} \frac{x}{x+y+z} = 1$$

Therefore,

$$\sum_{\text{cyc}} \frac{1}{x+2y+6} \leqslant \sum_{\text{cyc}} \frac{x}{\sqrt{x^2+4\sqrt{y}+4\sqrt{z}}}$$

as desired. □

Solution 2. (Pitchayut Saengrungkongka)

We provide a more direct way in proving both inequalities in Solution 1. For the left inequality, as in Solution 1, it suffices to show that

$$\sum_{\text{cyc}} \frac{1}{x+2y+6} \leqslant \frac{1}{3}$$

Clearing denominators and expanding everything. This is equivalent to

$$\begin{aligned}
& 3\sum_{\text{cyc}}(x+2y+6)(y+2z+6) \leqslant (x+2y+6)(y+2z+6)(z+2x+6) \\
\iff\ & 324+\sum_{\text{cyc}}(21xy+6x^2+108x) \leqslant 216+9xyz+\sum_{\text{cyc}}(2x^2y+4y^2x+42xy+12x^2+108x)
\end{aligned}$$

Or after cancellations and using $xyz=1$,

$$\sum_{\text{cyc}}(2x^2y+4y^2x+21xy+6x^2) \geqslant 99$$

which is clear by AM-GM'ing each term.

For the right inequality, substitute $x = a^2, y = b^2, z = c^2$. Thus $abc = 1$. Moreover by Holder's inequality

$$\left(\sum_{\text{cyc}} \frac{1}{\sqrt{a^4+4b+4c}}\right)^{\frac{1}{2}} \left(\sum_{\text{cyc}} a^2(a^4+4b+4c)\right)^{\frac{1}{4}} \left(\sum_{\text{cyc}} a^2\right)^{\frac{1}{4}} \geqslant a^2+b^2+c^2$$

So we need to prove that

$$\begin{aligned}
&\sum_{\text{cyc}} a^2(a^4+4b+4c) \leqslant (a^2+b^2+c^2)^3 \\
\iff &\sum_{\text{cyc}} (a^6+4a^3b^2c+4a^3bc^2) \leqslant 6a^2b^2c^2 + \sum_{\text{cyc}} (a^6+3a^4b^2+3a^2b^4) \\
\iff &\sum_{\text{cyc}} 4a^3b^2c+4a^3bc^2 \leqslant 6a^2b^2c^2 + \sum_{\text{cyc}} (3a^4b^2+3a^2b^4) \\
\iff &\sum_{\text{cyc}} (3a^2b^2+3a^2c^2+2a^2bc)(b-c)^2 \geqslant 0
\end{aligned}$$

which is true so we are done. □

Problem 2. Let $n < 2017$ be a fixed positive integer. Exactly n of the vertices of a regular 2017-gon are coloured red, and the other vertices are coloured blue. Prove that the number of isosceles triangles which vertices are the same color only depend on n, but not the configuration of red and blue points.

Solution 1. (Nitit Jongsawatsataporn)

We claim that the desired number is $\frac{3}{2}n(2017-n)$. To see this, pick one red vertex and one blue vertex, use these two vertices to construct isosceles triangle in every possible ways. Since $\gcd(2017, 6) = 1$, there are exactly three ways to construct isosceles triangle using this process but each isosceles triangle can be obtained in two different ways using this process. Hence our claim is true and we are done. □

Problem 3. Do there exists an arithmetic progression with 2017 terms which each term is not perfect power but the product of all 2017 terms is perfect power?

Solution 1. (Sithipont Cholsaipant)

Such sequence exists. For convenience, let $p = 2017$ which is a prime. We claim that the sequence $p! \cdot 1, p! \cdot 2, ..., p! \cdot p$ works.

First, if $n < p$ then $\nu_p(p! \cdot n) = 1$ therefore $p! \cdot n$ is not a perfect power. Furthermore, since 2011 is prime and $\nu_{2011}(p \cdot p!) = 1$, $p \cdot p!$ is also not perfect power.

Finally, note that the product of all terms is

$$(p! \cdot 1)(p! \cdot 2)...(p! \cdot p) = (p!)^{p+1}$$

which is clearly perfect power hence we are done. $\square$

Solution 2. (Thammadole Tansriwararat)

By Green-Tao Theorem, there exists an arithmetic progression $p_1, p_2, ..., p_{2017}$ consisting only primes. Let $P = p_1 p_2 ... p_{2017}$ and take $a_i = Pp_i$ for each i. It's easy to see that each term is not a perfect power, but the product of all terms is

$$(Pp_1)(Pp_2)...(Pp_{2017}) = P^{2018}$$

which is perfect power so we are done. $\square$

Problem 4. Let ΔABC be an acute triangle with altitudes AA_1, BB_1, CC_1 and orthocenter H. Let K, L be the midpoints of BC_1, CB_1. Let ℓ_A be the external angle bisector of $\angle BAC$. Let ℓ_B, ℓ_C be the lines through B, C and perpendicular to ℓ_A. Let ℓ_H be the line through H, parallel to ℓ_A. Prove that the centers of circumcircle of $\Delta A_1B_1C_1, \Delta AKL$ and the rectangle formed by $\ell_A, \ell_B, \ell_C, \ell_H$ are colinear.

Solution 1. (Pitchayut Saengrungkongka)

Let $P = \ell_A \cap \ell_B,\ Q = \ell_A \cap \ell_C,\ R = \ell_H \cap \ell_B,\ S = \ell_H \cap \ell_C$. Let M be the midpoint of BC and I be the incenter of ΔABC. We claim that circles $\odot(A_1B_1C_1), \odot(AKL), \odot(PQRS)$ pass through A_1 and M which will immediately imply the problem.

Note that the first circle is nine-point circle of ΔABC which is known to pass through M.

For the second circle, note that $MB = MC = MB_1 = MC_1$, therefore $MK \perp AB, ML \perp AC$. Hence $\angle AKM = \angle ALM = 90^\circ = \angle AA_1M$ which implies A_1, M pass through $\odot(AKL)$ as claimed.

Now we restrict attention to proving $PQRSA_1M$ is cyclic. To prove that PQA_1M is cyclic, let N be the midpoint of PQ. Therefore $MN \perp PQ$ implying MN is the perpendicular bisector of PQ. Hence $MP = MQ$. Now note that

$$\begin{aligned}(AI \cap BC, \ell_A \cap BC; B, C) = -1 &\implies (A, \ell_A \cap BC; P, Q) = -1 \\ &\implies (A_1A, A_1M; A_1P, A_1Q) = -1\end{aligned}$$

But $\angle AA_1M = 90^\circ$, this implies that A_1M is the external angle bisector of $\angle PA_1Q$. Combining with $MP = MQ$ gives PQA_1M is cyclic.

Now we prove that RSA_1M is cyclic. To do that, it suffices to show that ℓ_H bisects $\angle BHC$ so that we can repeat the same logic as above. Now note that

$$\measuredangle(\ell_H, BH) = \measuredangle(\ell_A, AC) + \measuredangle(AC, BH) = -\measuredangle(\ell_A, AB) - \measuredangle(AB, CH) = -\measuredangle(\ell_H, CH)$$

implying the desired bisection. Now suppose that circles $\odot(PQA_1M), \odot(RSA_1M), \odot(PQRS)$ are different. By radical axis theorem on these circles, lines PQ, RS, A_1M are concurrent which is absurd since $PQ \parallel RS \nparallel A_1M$. Hence $PQRSA_1M$ is concyclic implying the conclusion. □

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§3.2 Day 2 Solutions

Problem 1. Determine all positive integer n such that for any positive integers $a_1, a_2, ..., a_n$ which their sum is not divisible by n, there exists indices i such that each of

$$a_i,\ a_i + a_{i+1},\ a_i + a_{i+1} + a_{i+2}, ...,\ a_i + a_{i+1} + ... + a_{i+n-1}$$

is not divisible by n where $a_i = a_{i-n}$ for any $i > n$.

Solution 1. (Pitchayut Saengrungkongka)

The answer is all prime number. To show that composite n does not work, take a prime factor p of n and observe that

$$(a_1, a_2, ..., a_n) = \left(0, \frac{n}{p}, \frac{n}{p}, ..., \frac{n}{p}\right)$$

does not satisfy the problem's condition. Now we proceed that all prime p works. Assume that the sequence $(a_1, a_2, ..., a_n)$ does not obey the problem's condition. Therefore for any i, we can find integer $f(i) \in [i+1, i+p]$ such that

$$p \mid a_i + a_{i+1} + ... + a_{f(i)-1}$$

Consider $1, f(1), f(f(1)), ..., f^p(1)$. Observe that these integer are in $[1, p^2]$. By Pigeonhole's principle, we can find integers $0 \le a < b \le p$ such that

$$p \mid f^b(1) - f^a(1)$$

. Now by the definition of f,

$$p \mid \sum_{i=f^a(1)}^{f^{a+1}(1)-1} a_i + \sum_{i=f^{a+1}(1)}^{f^{a+2}(1)-1} a_i + \sum_{i=f^{a+2}(1)}^{f^{a+3}(1)-1} a_i + ... + \sum_{i=f^{b-1}(1)}^{f^b(1)-1} a_i$$

since p divides each of the summation. This implies that

$$p \mid \sum_{i=f^a(1)}^{f^b(1)-1} a_i = \frac{f^b(i) - f^a(i)}{p}(a_1 + a_2 + ... + a_p)$$

Therefore $p^2 \mid f^b(i) - f^a(i)$ but $0 < f^a(i) < f^b(i) \leq p^2$, contradiction. □

Problem 2. For any finite set $M \subset \mathbb{Z}^+$ and set $A \subseteq M$, define

$$f_M(A) = \{x \in M | x \text{ is divisible by odd number of elements of } A\}$$

For any positive integer k, we call M k-colorable if and only if every subsets of M can be assigned one of k colours so that for any $A \subseteq M$ such that $f_M(A) \neq A$, sets $f_M(A)$ and A must be different color.

Determine the least positive integer k such that every finite set $M \subset \mathbb{Z}^+$ is k-colorable.

Solution 1. (Linear Algebra, Krit Boonsiriseth)

Obviously, one color does not suffices. We claim that two colors suffices. Take the obvious graph interpretion G where each vertex represents subsets $A \subseteq M$ and all edges connect pair in form $\{A, f_M(A)\}$. It suffices to prove that G has no odd cycles.

Let $n = |M|$ and for convenience, denote f by f_M. Let $M = \{x_1, x_2, ..., x_n\}$ where $x_1 > x_2 > ... > x_n$ Construct the $n \times n$ matrix $\mathbf{T} = [t_{ij}]$ over $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ which

$$t_{ij} = \begin{cases} 0 & \text{where } x_i \nmid x_j \\ 1 & \text{where } x_i \mid x_j \end{cases}$$

Observe that all entries in the main diagonal are 1 and all entries below the main diagonal are 0. We also represent A by row vector (an $1 \times n$ matrix) $\mathbf{A} = [a_i]$ where

$$a_i = \begin{cases} 0 & \text{where } x_i \notin A \\ 1 & \text{where } x_i \in A \end{cases}$$

The merit of this representation is by using matrix multiplication, we can check that in $\mathbb{Z}_2$, we have

$$f(\mathbf{A}) = \mathbf{AT}$$

Armed with this interpretation, we will prove the following two claims.

Claim 1. f is injective

Proof. First, we prove that $\mathbf{T}$ has inverse $\mathbf{T}^{-1}$ by doing row operation.

The only 1 in the first row is at t_{11} so we can use this row to eliminate all 1's in the first column. Now the only 1 in the second column is at t_{22} so we can repeat this process. This process continues until the matrix $\mathbf{T}$ is an identity matrix. So $\mathbf{T}$ has inverse.

Now suppose that $f(\mathbf{A}) = f(\mathbf{B})$ for some row vectors $\mathbf{A}, \mathbf{B}$, we get

$$\mathbf{AT} = \mathbf{BT} \implies \mathbf{ATT}^{-1} = \mathbf{BTT}^{-1} \implies \mathbf{A} = \mathbf{B}$$

as claimed. □

By Claim 1, graph G is a union of disjoint cycles.

Claim 2. Let $k = 2^{\binom{n}{2}}$, then $f^k(\mathbf{A}) = \mathbf{A}$ for any row vector $\mathbf{A}$.

Proof. Observe that $f^k(\mathbf{A}) = \mathbf{T}^k\mathbf{A}$. Thus it suffices to show that $\mathbf{T}^k = \mathbf{I}$. Note that the set of all $n \times n$ matrices $\mathbf{M} = [m_{ij}]$ with the following properties

- $m_{ii} = 1$ for any $i \in \{1, 2, ..., n\}$
- $m_{ij} = 0$ for any $1 \leqslant i < j \leqslant n$

form a group G with matrix multiplication. (It can be checked through linear map definition of matrix.) Moreover $|G| = k$ hence by LAGRANGE'S THEOREM, $\mathbf{T}^k = \mathbf{I}$ as desired. □

Now Claim 2 implies length of any cycle must divides $k = 2^{\binom{n}{2}}$ so length of any cycle must be power of two. Since there is no cycle with length 1, there is no odd cycles as desired. □

Problem 3. Let S be a finite set and let $f : S \to S$ be a function. Suppose that for any function $g : S \to S$ not the same as f, functions $f \circ g \circ f$ and $g \circ f \circ g$ are different. Prove that any elements of S which is in the range of f must be in the range of $f \circ f$.

Solution 1. (Papon Lapate)

Define $S_1 = S$ and $S_{i+1} = Im(S_i)$ and let n be the smallest integer such that $S_n = S_{n+1}$.

Now, let $T_1 = S_n$ and define T_{i+1} to be the pre-image of T_i. It is easy to see that $S_i \subseteq T_i$, thus there exists smallest m such that $T_m = S$. Finally let $T'_1 = T_1$ and $T'_{i+1} = T_{i+1} - T_i$. It is suffice to show that $m \leqslant 2$.

First, since $Im(T'_1) = T'_1$, there must exists bijection $h : T'_1 \to T'_1$ such that $f(x) = h(x)$ for all $x \in T'_1$.
Now, suppose that $m > 2$, then we define g as follow:

- if $x \in T_1' \cup T_2'$, then let $g(x) = f(x)$.
- if $x \in T_{i+2}'$, then let $g(x) = h^{-i}(f^{i+1}(x))$.

It is easy to verify (and thus left as an exercise to the readers) that g is valid, not equivalent to f and satisfied the condition $f \circ g \circ f = g \circ f \circ g$ which is a contradiction. Hence $m \leqslant 2$ and we are done. □

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§3.3 Day 3 Solutions

Problem 1. Rectangle $\mathcal{R}$ which its sides are positive odd integer is tiled by some small rectangles, each of them has integer sides. Prove that there is one of those small rectangles which the four distances from one of its sides to the closest side of $\mathcal{R}$ are all odd or even.

Solution 1. (Pitchayut Saengrungkongka)

Fix all corners of $\mathcal{R}$ to be $(0,0),(m,0),(n,0),(m,n)$. It's easy to see that all vertices of smaller rectangles are lattice point.

Imagine $\mathcal{R}$ as the $m \times n$ board. Color $\mathcal{R}$ in chess order so that all corners are black. It's easy to see that all rectangles which satisfy the problem's condition are precisely all rectangles which all four corners is black or equivalently, all rectangle which has black cells more than white cells. But $\mathcal{R}$ has this property too hence we are done. □

Problem 2. Let O, H be the circumcenter and orthocenter of triangle ABC. Select points P, Q on the line AO such that $BP \perp AC$ and $CQ \perp AB$. Prove that circumcenter of triangle PQH lies on one of the medians of triangle ABC.

Solution 1. (Nithid Anchaleenukoon)

Let S be circumcenter of ΔPQH and M be midpoint of BC and D, E, F are feet of altitudes from A, B, C in ΔABC. Angle chasing reveals that

$$\angle HPQ = \angle EPA = 90^\circ - \angle OAC = 90^\circ - \angle BAH = \angle B$$

But $\angle DHC = \angle B$ so circumcircle of PQH tangent to AH so $HS \parallel BC$.

Now let S' be intersection point of AM and line perpendicular to AH at H. The homothety centered at A taking H to D will send S', P, E to M, P', E' repectively, where E' is point

BC such that $DE' \perp AC$ and $P' = DE' \cap AO$. It's now suffices to prove that $DM = MP'$.

Because $\angle E'P'A = \angle EPA = \angle B$. So A, B, D, P', E are concyclic.

By angle chasing, $BE \parallel DP'$, $BP'E = \angle EBC$ and $\angle BEM = \angle EBC$. So E', P, M collinear. But $MB = ME$ and $BE \parallel DP$, we are done. $\square$

Solution 2. (Pitchayut Saengrungkongka)

Let D, E, F be feet of altitudes from A, B, C of ΔABC respectively. Let $R = EF \cap BC$ and let M be the midpoint of BC. Let ω be the circumcircle of ΔPQH which is centered at S.

By angle chasing, AH is tangent to ω (all angles in ΔPQH are easy to find). Now let AK be other tangent to ω and note that by projection at H

$$-1 = (H, K; P, Q) = (D, HK \cap BC; B, C)$$

but $(D, R; B, C) = -1$, therefore H, K, R are colinear. By Brokard's theorem on $BCEF$, $HR \perp AM$. Therefore $HK \perp AM$. But $AS \perp HK$ hence we are done. $\square$

Problem 3. Determine the smallest positive integer n such that there exists infinitely many n-tuple of positive rational numbers $(a_1, a_2, ..., a_n)$ such that

$$a_1 + a_2 + ... + a_n,\ \frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}$$

are integer.

Solution 1. The answer is $n = 3$. We first prove that $n = 2$ fails. Let $a_1 = \frac{a}{b}, a_2 = \frac{c}{d}$ where $\gcd(a, b) = \gcd(c, d) = 1$. Note that

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \in \mathbb{Z}$$

therefore $b \mid ad + bc \implies b \mid ad \implies b \mid d$. Similarly $d \mid b$ hence $b = d$. Since $\frac{b}{a} + \frac{d}{c} \in \mathbb{Z}$, repeating similar logic gives $a = c$ therefore $a_1 = a_2$. It can be easily checked now that there are only finitely many such tuples.

Now we proceed to give the construction for $n = 3$. Recall that the equation $r^2 + s^2 + 1 = 3rs$ has infinitely solution $r, s \in \mathbb{Z}^+$ as $(1, 2)$ is one of them and Vieta tells us that if (r, s) is solution then so is $(s, 3s - r)$. Now take

$$a_1 = \frac{1}{rs},\ a_2 = \frac{r}{s} + 1,\ a_3 = \frac{s}{r} + 1$$

It's easy to check that $a_1 + a_2 + a_3 = 5$ and $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = rs + 1$ which are both integer. Hence we are done. $\square$

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§3.4 Day 4 Solutions

Problem 1. Let $a_1, a_2, ..., a_n$ be positive integers, not all equal to 1. Let $M = a_1a_2...a_n$ and $k = \frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}$. Suppose that k is integer. Prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive real roots.

Solution 1. (Jirayus Jinapong)

Assume that the polynomial $P(x)$ has a positive real root r. By Bernoulli Inequality, we get

$$(r+1)^{\frac{1}{a_i}} \geqslant 1 + \frac{r}{a_i} \implies a_i(r+1)^{\frac{1}{a_i}} \geqslant (r+a_i)$$

Thus,

$$\prod_{i=1}^{n} a_i(r+1)^{\frac{1}{a_i}} \geqslant \prod_{i=1}^{n}(r+a_i) \implies M(r+1)^k - \prod_{i=1}^{n}(r+a_i) \geqslant 0$$

The equality holds if and only if $a_1 = a_2 = ... = a_n = 1$, which contradicts the problem's condition. Thus

$$M(r+1)^k > (r+a_1)(r+a_2)...(r+a_n) \implies P(r) > 0,$$

impossible. Hence, the polynomial $P(x)$ has no positive real roots as desired. □

Problem 2. Let $(x_1, x_2, ..., x_{100})$ be a permutation of $\{1, 2, ..., 100\}$. Define

$$S = \{m \mid m \text{ is the median of } \{x_i, x_{i+1}, x_{i+2}\} \text{ for some } i\}.$$

Determine the minimum possible value of the sum of all elements of S.

Solution 1. (Jirayus Jinapong)

The answer is $\boxed{1122} = 2 + 4 + 6 + ... + 66$, which can be archived by the permutation.

$$((100,\ 2,\ 1),\ (99,\ 4,\ 3),\ (98,\ 6,\ 5),\ ...,\ (68,\ 66,\ 65),\ 67).$$

Now we are left to prove the bound. Define $\text{med}\{a,b,c\}, \min\{a,b,c\}$ as the median and the lowest number of $\{a,b,c\}$ respectively. Let $y_i = \text{med}\{x_{3i-2}, x_{3i-1}, x_{3i}\}$ and $t_i = \min\{x_{3i-2}, x_{3i-1}, x_{3i}\}$ for each $i = 1, 2, ..., 33$.

It's obvious that $t_i \neq y_j, i, j \in \{1, 2, ..., 33\}$ and the desired sum is at least $y_1 + y_2 + ... + y_{33}$ Let $z_1, z_2, ..., z_{33}$ be the permutation of $y_1, y_2, ..., y_{33}$ in ascending order. We claim that

Claim: $z_i \geqslant 2i$ for any $i = 1, 2, ..., 33$.

Proof: It suffices to show that there are at least $2i - 1$ numbers less than z_i. But

$$z_1, z_2, ..., z_{i-1} < z_i \text{ and } t_1, t_2, ..., t_i < z_i$$

hence $z_i \geqslant 2i$ as claimed.

Hence, the sum of element in S is at least $2 + 4 + ... + 66 = 1122$ as desired. □

Problem 3. Let $ABCC_1B_1A_1$ be a convex hexagon such that $BA = BC$ and lines AA_1, BB_1 and CC_1 share the same perpendicular bisector. Let ω denotes circumcircle of ΔABC. Diagonals AC_1 and A_1C meet at D. Circumcircles of triangle A_1BC_1 intersects ω at $E \neq A$. Prove that lines BB_1 and DE intersects on ω.

Solution 1. (Pitchayut Saengrungkongka)

We begin with some easy observations. Let ℓ be the common perpendicular bisector and O be the center of ω. Clearly $D \in \ell$. By Radical Axis on $\odot(A_1BC_1), \odot(ABC), \odot(ACA_1C_1)$, lines BE, AC, A_1C_1 are concurrent at T. By symmetry, $T \in \ell$.

Now let $F = BB_1 \cap \omega$ and let $D' = EF \cap \ell$. We have to prove that $D' = D$, which we will do by showing that ℓ externally bisects $\angle BD'C$.

Consider the circle $\gamma = \odot(ED'T)$. This circle is orthogonal to ω because

$$\angle OED' = 90^\circ - \angle EBF = \angle BTD'$$

thus OE touches γ, implying the claimed orthogonality. Since T is foot of external bisector of $\angle AEC$, circle γ must be E-Apollonius circle of ΔAEC. Since $D' \in \gamma$, we get

$$\frac{AD'}{D'C} = \frac{AT}{TC} \implies D'T \text{ externally bisects } \angle AD'C$$

so we are done. □

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§3.5 Day 5 Solutions

Problem 1. Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$ and $\angle EDC = \angle ABC$. Prove that lines AC, BD and the line from E perpendicular to BC are concurrent.

Solution 1. (Pitchayut Saengrungkongka)

Let $P = AB \cap CD,\ Q = BC \cap ED,\ R = BC \cap EA,\ T = AC \cap BD$. The angle conditions imply that $BDQP$ and $CARP$ is cyclic. Combining with $AB = BC = CD$ gives $PC = CQ$ and $PB = BR$.

Since $AB = BC$, line AT is parallel to angle bisector of $\angle PBC$ so if we let I be the incenter of ΔPBC, we get that $BICT$ is parallelogram. Thus the foot K from T to BC is the contact point of P-excircle of ΔPBC.

Since $\angle EQR = \angle BPC = \angle ERQ$, we get that ΔEQR is isosceles. Furthermore, length chasing reveals that

$$QK = QC + CK = PC + \frac{PB + BC - PC}{2} = \frac{QR}{2}$$

hence $QK = KR \implies EK \perp QR$. But $TK \perp QR$ so we are done. □

Problem 2. Determine all ordered pairs (p, q) of prime numbers which $p > q$ and

$$\frac{(p+q)^{p+q}(p-q)^{p-q} - 1}{(p+q)^{p-q}(p-q)^{p+q} - 1}$$

is an integer.

Solution 1. (Pitchayut Saengrungkongka)

The answer is $(p, q) = (3, 2)$ which is easy to verify. Now we restrict attention to proving that there are only one. Manipulate the divisibility as

$$\begin{aligned} & (p+q)^{p-q}(p-q)^{p+q} - 1 \mid (p+q)^{p+q}(p-q)^{p-q} - (p+q)^{p-q}(p-q)^{p+q} \\ \iff & (p+q)^{p-q}(p-q)^{p+q} - 1 \mid (p+q)^{p-q}(p-q)^{p-q}((p+q)^{2q} - (p-q)^{2q}) \\ \iff & (p+q)^{p-q}(p-q)^{p+q} - 1 \mid (p+q)^{2q} - (p-q)^{2q} \end{aligned}$$

Thus we obtain the bound $p \leqslant 3q$, which reduce the case $q = 2$ to a finite case-check. From now, assume that p, q are odd. Let $a = p + q, b = p - q$. We have the following claim.

Claim. Every prime divisor r of $a^b b^a - 1$ must be in form $qk + 1$ unless $p \equiv \pm 1 \pmod q$.

Proof. We have

$$a^{2q} \equiv b^{2q} \pmod r \implies d = \operatorname{ord}_r \left(\frac{a}{b}\right) = 1, 2, q, 2q$$

If $d = 1, 2$, then $r \mid a^2 - b^2 = 4pq$ so $r = 2, p, q$. But $2 \nmid a^b b^a - 1$ so $r = p, q$.

- If $r = p$, then
$$p \mid a^b b^a - 1 \implies q^a(-q)^b - 1 \equiv 0 \pmod{p} \implies p \mid q^{2q} - 1$$
so $d = \operatorname{ord}_p(q) = 1, 2, q, 2q$. But $p > q \implies p \nmid (q-1)(q+1)$ so $q \mid d$. Hence $q \mid p - 1 \implies p \equiv 1 \pmod{q}$.

- If $r = q$, then
$$q \mid a^b b^a - 1 \implies q \mid p^b p^a - 1 \implies q \mid p^{2p} - 1$$
so $d = \operatorname{ord}_q(p) = 1, 2, p, 2p$. But $d \leqslant p - 1 < q$ so $d = 1, 2$. Hence $q \mid p^2 - 1$ implying $p \equiv \pm 1 \pmod{q}$.

If $d = q, 2q$, then since $d \mid r - 1$, we have $r \equiv 1 \pmod{q}$ as claimed. □

Now suppose that every prime divisor r of $a^b b^a - 1$ is in form $qk + 1$, notice that
$$a^b b^a - 1 = \left(a^{\frac{b}{2}} b^{\frac{a}{2}} - 1\right)\left(a^{\frac{b}{2}} b^{\frac{a}{2}} + 1\right).$$
Thus each factor must congruent to $1 \pmod{q}$, which is impossible.

Thus we must have the edge case $p \equiv \pm 1 \pmod{q}$. By the bound $p \leqslant 3q$, this reduces to $p = 2q \pm 1$.

- If $p = 2q - 1$, then we have
$$(3q-1)^q (q-1)^{3q-1} - 1 \mid (3q-1)^{2q} - q^{2q}$$
We have
$$(q-1)^3 > 2(3q-1) \implies (q-1)^{3q} > 2^q (3q-1)^q > (q-1)(3q-1)^q$$
so $(q-1)^{3q-1} > (3q-1)^q$, hence the left hand side is bigger, implying contradiction.

- If $p = 2q + 1$, then we have
$$(3q+1)^q (q+1)^{3q+1} - 1 \mid (3q+1)^{2q} - q^{2q}$$
Using some calculus, we can show that $q^{3q+1} > (3q+1)^q$ thus the left hand side is always bigger, this yields contradiction.

Having exhausted all cases, the only answer is $(3, 2)$ as desired. □

Problem 3. Let n be a positive integer. Consider an $n \times n \times n$ cube, which each of n^3 cubes is colored. Define a *box* as an $n \times n \times 1$ grid of cubes, in any of three possible orientation. We also define the *color-set* of each box as the set of all color. Suppose that for any box

$\mathcal{B}_1$, there exists another two boxes $\mathcal{B}_2, \mathcal{B}_3$ such that $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ have different orientations but having same color-set. Determine the maximum possible number of colors used.

Solution 1. (Author: Jirayus Jinapong Co-author: Nitit Jongsawatsataporn)

We claim that the answer is $\sum_{i=1}^{n} i^2$

The problem is divided into 2 main parts.

Part 1: Bound the number of maximum color

The trick is to generalize the problem by allow the *invisible color*, which has the following properties.

- It can be assigned to any cube. If it is assigned to a cube, then that cube cannot be assigned any color anymore.
- It counts toward neither the total number of colors nor the color set of a box.

Now the induction is amusingly trivial. Clearly the $1 \times 1 \times 1$ cube use at most one color. Suppose that we color an $(n+1) \times (n+1) \times (n+1)$ cube which satisfies the generalized problem's condition. We can find three boxes $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$, having different orientation and having the same coloring set.

Now remove those three boxes, leaving with an $n \times n \times n$ cube. Furthermore, change the color of any unit cube which its color appeared in $\mathcal{B}_1$ to the invisible color. We lose at most $(n+1)^2$ colors. Moreover, the new cube satisfies the generalized problem's condition so the $n \times n \times n$ cube uses at most $1^2 + 2^2 + ... + n^2$ cubes as desired.

Part 2: Construction of the cube

We'll find the construction by induction. Let $P(n)$ denote the assertion "There exists a cube that has exactly $\sum_{i=1}^{n} i^2$ colors."

Base Case: is obvious because we can just give arbitrary color to the unit cube.

Induction Step: Assume that $P(n)$ is true for some natural number $n \geqslant 1$. We'll show that there exists a $(n+1) \times (n+1) \times (n+1)$ cube satisfy the condition. Consider a $n \times n \times n$ cube $\mathcal{C}$ that satisfy the condition.

First, give the coordinates (i, j, k) to a unit cube which the i^{th}, j^{th}, k^{th} cube from the side, front, and top, respectively.

Without loss of generality, assume that the i^{th} block in each orientation has the same color-set (The i^{th} block from the side, front, top orientation is the i^{th} when count from the left side, front, and top of the cube).

We'll add $3n^2 + 3n + 1$ cube to the first $n \times n \times n$ cube to be $(n+1) \times (n+1) \times (n+1)$ cube under the following procedure.

1. Paint $(n+1, n+1, n+1)$ with color a_1 which is not appear in $\mathcal{C}$.
2. Paint $(n+1, i, i)$ and $(i, n+1, n+1)$ with color a_{i+1} which are not appear in $\mathcal{C}$,
3. Paint $(i, n+1, i)$ and $(n+1, i, n+1)$ with color a_{i+n+1} which are not appear in $\mathcal{C}$
4. Paint $(i, i, n+1)$ and $(n+1, n+1, i)$ with color a_{i+2n+1} which are not appear in $\mathcal{C}$
5. Paint $(n+1, i, j), (j, n+1, i), (i, j, n+1)$ with the same color total $n^2 - n$ colors which are not appear in $\mathcal{C}$.

It's not hard to show that the new cube satisfy the condition.
Hence, $P(n+1)$ is true.
By Induction Principle, $P(n)$ is true for all natural number n.
From two parts, the maximum color is $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. □

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§3.6 Day 6 Solutions

Problem 1. Let ABC be a triangle with points E, F lies on segment BC. Let K, L be points on segments AB, AC respectively such that $EK \parallel AC$, $FL \parallel AB$. The incircle of triangles BEK, CFL touches segments AB, AC at X, Y respectively. Lines AC and EX intersect at M and lines AB and FY intersects at N. Suppose that $AX = AY$. Prove that $MN \parallel BC$.

Solution 1. (Pitchayut Saengrungkongka)

Since $\Delta XKE \sim \Delta XAM$ and $\Delta YLF \sim \Delta YAN$, we get

$$\begin{aligned}\frac{AM}{AN} &= \frac{AM}{KE} \cdot \frac{LF}{AN} \cdot \frac{KE}{LF} \\ &= \frac{XA}{XK} \cdot \frac{YL}{AY} \cdot \frac{KE}{LF} \\ &= \frac{KE}{KX} \cdot \frac{LY}{LF}.\end{aligned}$$

Let the incircle of ΔBKE touches KE at Z so $\Delta BKE \cup Z \sim \Delta FLC \cup Y$. Furthermore both triangles are similar to ΔABC. Hence

$$\frac{AM}{AN} = \frac{KE}{KZ} \cdot \frac{LY}{LF} = \frac{LC}{LY} = \frac{AB}{AC}$$

implying the conclusion. □

Problem 2. Call a rational number *short* if it has terminating decimal. For a positive integer m, call a positive integer t m-*tastic* if and only if there exists $c \in \{1, 2, ..., 2017\}$ which $\frac{10^t-1}{cm}$ is short but $\frac{10^k-1}{cm}$ is not short for any positive integer $k < t$.

Let $S(m)$ be the number of all m-tastic integer. Determine the maximum possible value of $S(m)$ over all positive integer m.

Solution 1. (Pitchayut Saengrungkongka)

The answer is $\boxed{807}$. First, we can made the assumption that $\gcd(m, 10) = 1$. Call the number c as *chef* of t. For convenience, let $\mathcal{C} = \{n \in \mathbb{Z}^+ \mid n \leqslant 2017,\ \gcd(n, 10) = 1\}$

Note that if c is a chef of t, then $c' = \frac{c}{2^a 5^b}$ is also a chef of t. Thus WLOG, we can assume that $\gcd(c, 10) = 1$.

Now observe that if $c \in \mathcal{C}$ is a chef of t then $\gcd(cm, 10) = 1$ so

$$cm \mid 10^t - 1 \text{ but } cm \nmid 10^k - 1 \text{ for any } k < t \implies t = \text{ord}_{cm}(10).$$

Thus each $c \in \mathcal{C}$ is a chef of a unique tastic number t. Hence $S(m) \leqslant |\mathcal{C}| = 807$claimed.

Now we have to construct m which $\text{ord}_{cm}(10)$ are all distinct for each $c \in \mathcal{C}$. To do that, we enumerate primes $p_1 < p_2 < ... < p_k = 2017$, disregarding $2, 5$. And define

$$a_i = \nu_{p_i}(10^{p_i - 1} - 1) \text{ for each } i = 1, 2, ..., k$$

By Lifting The Exponent Lemma, we easily find that

$$\nu_{p_i}(a^n - 1) = a_i + \nu_{p_i}(n) \implies \text{ord}_{p^t}(10) = {p_i}^{t - a_i}(p_i - 1)$$

for any $t \geqslant a_i$. Now for each $i = 1, 2, ..., k$, let $b_i = a_i + 2018$ and let $m = \prod_{i=1}^{k} {p_i}^{b_i}$. From discussions above, we get

$$\text{ord}_{cm}(10) = \text{lcm}\{{p_i}^{c_i + 2018}(p_i - 1)\}$$

where $c_i = \nu_{p_i}(c)$. Since $v_{p_i}(p_j - 1) < 2018$ for any $i, j \leqslant k$, we get

$$\nu_{p_i}(\text{ord}_{cm}(10)) = c_i + 2018 = \nu_{p_i}(c) + 2018$$

so for any prime $p \in \{p_1, p_2, ..., p_k\}$,

$$\nu_p(\text{ord}_{c_1 m}(10)) = \nu_p(\text{ord}_{c_2 m}(10)) \iff \nu_p(c_1) = \nu_p(c_2).$$

But this is true for arbitrary prime p which may divide some elements in $\mathcal{C}$ so the numbers $\{\text{ord}_{cm}(10) \mid c \in \mathcal{C}\}$ are distinct as desired. $\square$

Problem 3. Let $n \geqslant 3$ be an integer. Let $a_1, a_2, ..., a_n \in [0,1]$ which $a_1 + a_2 + \cdots + a_n = 2$. Prove that

$$\sqrt{1-\sqrt{a_1}} + \sqrt{1-\sqrt{a_2}} + \cdots + \sqrt{1-\sqrt{a_n}} \leqslant n - 3 + \sqrt{9 - 3\sqrt{6}}.$$

Solution 1. (Official Solution, Modified by Jirayus Jinapong)

Let $f : [0,1] \to [0,1]$ defined as $f(x) = \sqrt{1-\sqrt{x}}$. f is convex on $[0, \frac{4}{9}]$ and is concave on $[\frac{4}{9}, 1]$.

Consider the set

$$S = \left\{ \sum_{i=1}^{n} f(a_i) \mid \sum_{i=1}^{n} a_i = 2, \forall i, a_i \in [0,1] \right\}$$

Since S is a bounded compact subset of $\mathbb{R}$, it has a maximal element, which we will denote by M.

<u>Claim:</u> M must be of the form

$$M = (n+1-a) \cdot f(0) + f(2-at) + af(t)$$

for some integer $0 \leqslant a \leqslant n-1$ and some real t such that $t \in [\frac{4}{9}, 1]$ and $(2-at) \in [0, \frac{4}{9}]$.

Proof. First, we'll show that M has at most one $i \in \{1, 2, ..., n\}$ such that $a_i \in (0, \frac{4}{9})$. Assume the contrary, there exists two indices i, j such that $a_i, a_j \in (0, \frac{4}{9})$.

<u>Case 1:</u> $a_i + a_j < \frac{4}{9}$. WLOG, $a_i \geqslant a_j$. By Karamata's Inequality

$$(a_i + a_j, 0) \succ (a_i, a_j) \implies f(a_i) + f(a_j) \leqslant f(a_i + a_j) + f(0)$$

thus replacing (a_i, a_j) with $(a_i + a_j, 0)$ yields smaller value of M, contradiction.

<u>Case 2:</u> $a_i + a_j \geqslant \frac{4}{9}$. WLOG, $a_i \geqslant a_j$. By Karamata's Inequality,

$$\left(\frac{4}{9}, a_i + a_j - \frac{4}{9}\right) \succ (a_i, a_j) \implies f(a_i) + f(a_j) \leqslant f\left(\frac{4}{9}\right) + f\left(a_i + a_j - \frac{4}{9}\right)$$

thus replacing (a_i, a_j) with $\left(\frac{4}{9}, a_i + a_j - \frac{4}{9}\right)$ yields the smaller value of M, contradiction.

From two cases, M has at most one a_i such that $a_i \in \left(0, \frac{4}{9}\right)$. Next, we'll show that every a_i such that $a_i \geqslant \frac{4}{9}$ must have the same value.

Let $a_{i_1}, a_{i_2}, ..., a_{i_k} \in [\frac{4}{9}, 1]$ in descending order and $a = \frac{\sum_{j=1}^{k} a_{i_j}}{k}$. Then, by Karamata's Inequality,

$$(a_{i_1}, ..., a_{i_k}) \succ (a, ..., a) \implies \sum_{j=1}^{k} f(a_{i_j}) \leqslant kf(a).$$

Thus replacing $(a_{i_1}, ..., a_{i_k})$ with $(a, a, ..., a)$ yields the smaller value of M so every a_i which $a_i \geqslant \frac{4}{9}$ must have the same value. Hence we can conclude that M must be of the form

$$M = (n + 1 - a) \cdot f(0) + f(2 - at) + af(t)$$

. □

Back to the problem, it's obvious that $a \leqslant 4$.

<u>Case 1: $a = 4$</u> It suffices to show that, for $\frac{4}{9} \leqslant t \leqslant \frac{1}{2}$, we have

$$\sqrt{1 - \sqrt{2 - 4t}} + 4\sqrt{1 - \sqrt{t}} \leqslant 2 + \sqrt{9 - 3\sqrt{6}}.$$

It's direct to verify that the function $g_1(t) = \sqrt{1 - \sqrt{2 - 4t}} + 4\sqrt{1 - \sqrt{t}}$ is increasing on $\left[\frac{4}{9}, \frac{1}{2}\right]$. Thus, for $t \in \left[\frac{4}{9}, \frac{1}{2}\right]$, $g_1(t) \leqslant g_1(\frac{1}{2}) < 2 + \sqrt{9 - 3\sqrt{6}}$ as desired.

<u>Case 2: $a = 3$</u>, it suffices to show that, for $\frac{14}{27} \leqslant t \leqslant \frac{2}{3}$, we have

$$\sqrt{1 - \sqrt{2 - 3t}} + 3\sqrt{1 - \sqrt{x}} \leqslant 1 + \sqrt{9 - 3\sqrt{6}}.$$

It is direct to verify that $g_2(t) = \sqrt{1 - \sqrt{2 - 3t}} + 3\sqrt{1 - \sqrt{t}}$ is increasing on $\left[\frac{14}{27}, \frac{2}{3}\right]$. Thus, for $t \in \left[\frac{14}{27}, \frac{2}{3}\right]$, $g_2(t) \leqslant g_2(\frac{2}{3}) = 1 + \sqrt{9 - 3\sqrt{6}}$ as desired.

<u>Case 3: $a = 2$</u>, it suffices to show that, for $\frac{7}{9} \leqslant t \leqslant 1$, we have

$$\sqrt{1 - \sqrt{2 - 2t}} + 2\sqrt{1 - \sqrt{t}} \leqslant \sqrt{9 - 3\sqrt{6}}.$$

It is direct to verify that $g_3(t) = \sqrt{1 - \sqrt{2 - 2t}} + 2\sqrt{1 - \sqrt{t}}$ is decreasing on $\left[\frac{7}{9}, 1\right]$. Thus, for $t \in \left[\frac{7}{9}, 1\right]$, $g_3(t) \leqslant g_3(\frac{7}{9}) < \sqrt{9 - 3\sqrt{6}}$ as desired.

Having considered all the cases, we finish the proof. □

Remark 1. In Solution 1, There are many ways to carry out the single-variable inequality analyses (e.g. that $g_1(t) < 2 + \sqrt{9 - 3\sqrt{6}}$ for $\frac{4}{9} \leqslant t \leqslant \frac{1}{2}$. We omitted the proofs for these single-variable results here. However in order for a student to obtain full marks, s/he needs to provide complete proofs for these results.

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§3.7 Day 7 Solutions

Problem 1. Determine all functions $g : \mathbb{R} \to \mathbb{R}$ which there exists strictly monotone function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x)g(y) + f(y)$$

for all reals x, y.

Solution 1. (Nithid Anchaleenukoon)

Let $P(x,y)$ be the assertion $f(x+y) = f(x)g(y) + f(y)$.

$P(0,y) \implies f(0)g(y) = 0$. If $f(0) \neq 0$ then $g(y) = 0\ \forall y \in \mathbb{R} \implies f(x+y) = f(y)\ \forall x, y \in \mathbb{R}$, contradiction. Therefore $f(0) = 0$.

$P(1,0) \implies f(1) = f(1)g(0)$. But $f(1) \neq 0$ so $g(0) = 1$.

Since $P(y,x) \implies f(x+y) = f(y)g(x) + f(x)$. We get

$$f(x)g(y) + f(y) = f(y)g(x) + f(x) \qquad \forall x, y \in \mathbb{R}$$
$$(g(x)-1)f(y) = (g(y)-1)f(x) \qquad \forall x, y \in \mathbb{R}$$

Case 1. There exists $a \neq 0$ such that $g(a) = 1$.

So $f(a)(g(y)-1) = 0$. But $f(a) \neq 0 \implies g(y) = 1 \forall y \in \mathbb{R}$. There exist $f(x) = kx\ \forall x \in \mathbb{R}$ satisfy the problem.

Case 2. There don't exist $x \neq 0$ such that $g(x) = 1$. We get

$$\frac{g(x)-1}{f(x)} = \frac{g(y)-1}{f(y)} \qquad \forall x, y \in \mathbb{R} - \{0\}$$

and because $g(0) = 1$. So $g(x) = c \cdot f(x) + 1\ \forall x \in \mathbb{R}$ when $c = \frac{g(1)-1}{f(1)}$. Pluging this in $P(x,y)$ gives

$$f(x+y) = c \cdot f(x)f(y) + f(x) + f(y) \qquad \forall x, y \in \mathbb{R}$$
$$c \cdot f(x+y) + 1 = (c \cdot f(x) + 1)(c \cdot f(y) + 1) \qquad \forall x, y \in \mathbb{R}$$
$$g(x+y) = g(x)g(y) \qquad \forall x, y \in \mathbb{R}.$$

Since $g(x)$ is monotone, $g(x) = e^{kx}\ \forall x \in \mathbb{R}, k \neq 1$ and there exist $f(x) = \frac{e^{kx}-1}{c}\ \forall x \in \mathbb{R}$

Hence $g(x) = e^{kx}, k \in \mathbb{R}\ \forall x \in \mathbb{R}$ are all functions which satisfy the problem's condition. □

Problem 2. Sir Alex plays the following game in a row of 9 cells. At the beginning, all cells are empty. In each move, he performs exactly one of the following procedures.

i) He chooses one empty cells and inserts integer in form 2^k for some integer $k \geqslant 0$.

ii) He chooses two (not necessarily adjacent) cells which have the same value 2^k. He replaces one of the two cells with 2^{k+1} and erase the number in other cell.

Suppose that eventually, Sir Alex ends up with exactly one cells containing number 2^n for some positive integer n while all other cells are empty. Determine (in terms of n) the maximum possible number of moves he could have made.

Solution 1. (Pitchayut Saengrungkongka)

Replace 9 with arbitrary k. Let $f(n,k)$ denote the maximum number of moves to reach the situation where only 2^n left on the board.

Obviously $f(0,k) = f(n,1) = 1$. We prove the following recurrence.

Claim : $f(n,k) = f(n-1,k) + f(n-1,k-1) + 1$ for any n, k which $n \geqslant 2, k \geqslant 1$.

Proof. Obviously, we can make at least two moves. Thus the final 2^n must comes from using move 2 with $(2^{n-1}, 2^{n-1})$. We have to do subtasks $\mathcal{T}_1, \mathcal{T}_2$ of constructing 2^{n-1}.

The key observation is if we assume that $\mathcal{T}_1$ starts before $\mathcal{T}_2$, then $\mathcal{T}_1$ always take one cell to store numbers. Thus $\mathcal{T}_2$ can use at most $k-1$ cells. Hence all moves come from the following.

- The moves from $\mathcal{T}_1$, which is at most $f(n-1,k)$.
- The moves from $\mathcal{T}_2$, which is at most $f(n-1,k-1)$.
- The final move in combining two 2^{n-1}'s.

Hence $f(n,k) \leqslant f(n-1,k) + f(n-1,k-1) + 1$. Moreover, the equality holds if we finish $\mathcal{T}_1$ before starting $\mathcal{T}_2$. Thus $\mathcal{T}_2$ can use the remaining $k-1$ cells freely, implying the conclusion. □

Now we just have to solve the recurrence. To do that, we just simply guess and check. We claim that

$$f(n,k) = 2\left(\binom{n}{0} + \binom{n}{1} + ... + \binom{n}{k-1}\right) - 1.$$

This clearly satisfies the initial conditions $f(n,1) = f(0,k) = 1$, so we just have to prove that this satisfies the problem's condition. Note that

$$\begin{aligned} f(n-1,k) + f(n-1,k-1) + 1 &= 2\left(\sum_{i=0}^{k-1}\binom{n-1}{i} + \sum_{i=0}^{k-2}\binom{n-1}{i}\right) - 1 \\ &= 2\left(\sum_{i=0}^{k-1}\binom{n-1}{i} + \sum_{i=1}^{k-1}\binom{n-1}{i-1}\right) - 1 \\ &= 2\left(\sum_{i=0}^{k-1}\binom{n-1}{i} + \binom{n-1}{i-1}\right) - 1 \\ &= 2\left(\sum_{i=0}^{k-1}\binom{n}{i}\right) - 1 \\ &= f(n,k) \end{aligned}$$

Hence the answer is $2\left(\binom{n}{0} + \binom{n}{1} + ... + \binom{n}{8}\right) - 1$ and we are done. □

Problem 3. Let n be a fixed odd positive integer. For an odd prime p, define

$$a_p = \frac{\sum_{k=1}^{\frac{p-1}{2}} \left\{\frac{k^{2n}}{p}\right\}}{p-1}$$

Prove that a_p yields equal value for infinitely many primes p.

Notes : *For any real number x, define $\{x\} = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the greatest integer not exceeding x*

Solution 1. (Pitchayut Saengrungkongka)

We claim that $p \equiv 1 \pmod{4n} \implies a_p = \frac{1}{4}$ which will finish the problem by DIRICHLET'S THEOREM. First, we prove the following lemma.

Lemma : Equation $x^{2n} \equiv -1 \pmod{p}$ has a solution in $\pmod{p}$.

Proof. Let g be a generator $\pmod{p}$. Let $x = g^{\frac{p-1}{4n}}$, we get

$$x^{2n} \equiv g^{\frac{p-1}{2}} \equiv -1 \pmod{p}$$

as desired. □

Back to the main problem. Let $T = \{k^{2n} \pmod p \mid k = 1, 2, ..., p-1\}$. Since $2n \mid p-1$, it's well known that for any $t \in T$, the equation $x^{2n} \equiv t \pmod p$ has exactly $2n$ solutions in (mod p). Thus each element of T is counted n times in the multiset $\{1^{2n}, 2^{2n}, ..., \left(\frac{p-1}{2}\right)^{2n}\}$.

Moreover, $k^2 n \in T$ implies $(kx)^2 n \equiv -k^{2n} \pmod p$ where x is a solution from the Lemma. So $t \in T$ implies $p - t \in T$. Combining with above discussion, we get

$$\begin{aligned} a_p &= \frac{\sum_{k=1}^{\frac{p-1}{2}} k^{2n} \pmod p}{p(p-1)} \\ &= \frac{\frac{p}{2} \cdot \frac{p-1}{2}}{p(p-1)} \\ &= \frac{1}{4} \end{aligned}$$

as desired. □

Remark 1. As Solution 1 shows, the condition n odd is extraneous.

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§3.8 Day 8 Solutions

Problem 1. Let n be a fixed positive integer. Define a *chameleon* as any word of $3n$ letters which there are exactly n occurrences of letters a, b, c. Define a *swap* as a transposition of any two adjacent letters. Prove that for any chameleon X, there exists chameleon Y which cannot be reached from X using fewer than $3n^2/2$ swaps.

Solution 1. (Nithid Anchaleenukoon)

First, define an *inversion* is pair of letters which is one of the following types.

- A pair of (b, a) which b is on the left of a.
- A pair of (c, a) which c is on the left of a
- A pair of (c, b) which c is on the left of b.

Define $f : S \to \mathbb{N}$ where X is set of all chameleon with length $3n$ by $f(X)$ is number of inversion in chameleon X.

We see that if X swap one time,value of $f(X)$ will change exactly 1 (either increase or decrease). Because if $X = x_1x_2...x_{3n}$ swap x_i, x_{i+1},x_i, x_{i+1} are still on the right of $x_1x_2...x_{i-1}$ and on the left of $x_{i+2}, x_{i+3}...x_{3n}$.

So when X using fewer than $\frac{3n^2}{2}$ swaps to chameleon Y, $|f(X) - f(Y)| < \frac{3n^2}{2}$.

Define chameleons

$$M = \underbrace{aa...a}_{n}\underbrace{bb...b}_{n}\underbrace{cc...c}_{n}$$
$$N = \underbrace{cc...c}_{n}\underbrace{bb...b}_{n}\underbrace{aa...a}_{n}.$$

Since $f(M) = 0, f(N) = 3n^2$. It's easy to see that $0 \leqslant f(X) \leqslant 3n^2\ \forall X \in S$.So for every $X \in S$, there exist $Y \in \{M, N\}$ such that $|f(X) - f(Y)| \geqslant \frac{3n^2}{2}$.This mean there exist a chameleon Y which cannot be reached from X using fewer than $3n^2/2$ swaps. □

Problem 2. Let $a_1, a_2, a_3, ...$ be the sequence of real numbers satisfying

$$a_n = -\max_{i+j=n}(a_i + a_j) \quad \text{for any } n > 2017.$$

Prove that this sequence is bounded i.e., there exists real number M such that $|a_n| < M$ for any positive integer n.

Solution 1. (Nitit Jongsawatsataporn)

If this sequence contains only 0, this question is obvious. So we supposed this sequence is not constant. Define *local-maximum* as a value of n such that $|a_n| > |a_i| \forall i \in \mathbb{N}, i < n$. Call a local maximum n *positive* if $a_n > 0$ and *negative* if $a_n < 0$.

Lemma 1. There exist at most one positive local maximum beyond a_{2017}.

Proof. Since $n > 2017$ then there must exist i, j such that $a_n = -a_i - a_j$ and $i + j = n$ moreover $a_i + a_j$ is the highest value of any $i + j$. Supposed that there exists $m < n$ such that $a_m > 0$ then $a_n = a_n = -\max_{i+j=n}(a_i + a_j) < -(a_m + a_{n-m}) = -a_m - a_{n-m}$

Case 1 $a_{n-m} > 0$ then $a_n < 0$, contradiciton.

Case 2 $a_{n-m} < 0$ then $a_n = -a_m + |a_{n-m}| < |a_{n-m}|$ which contradicts the definition of local maximum. □

Lemma 2. If $a_n < 0$ and $n > 2017$ then $|a_n| < 2 \cdot \max_{i<n}(a_i)$.

Proof. This is pretty obvious. There exists k, l which $k + l = n$ and $a_n = -(a_k + a_l)$ hence

$$|a_n| = |a_k + a_l| \leq 2\max_{i<n}(a_i)$$

as desired. □

From lemma 1 we can choose either there exist *local-maximum* which is $a_n > 0$ or not. Let $m \in \mathbb{N}$ which is greater than *the* positive local maximum (if exists). Suppose for contradiction that this sequence is unbounded. We then define two more sequences.

- Define $m < t_1 < t_2 < \ldots$ as a sequence of (negative) local-maximum.
- Defined $m_1, m_2, \ldots$ be a sequence of m such that $a_{m_j} = \max_{i<t_j}(a_i)$

From Lemma 2, The second sequence is infinite, and cannot be stationary. So consider the lowest k such that $m_k > 2017$. Let l be the lowest number such that $a_{m_l} > a_{m_k}$. Obviously, $m_l > 2017$ so by the definition of sequence we get that

$$0 < a_{m_l} = -\max_{i+j=m_l}(a_i + a_j) \leqslant -(a_{m_k} + a_{m_l - m_k}) \implies a_{m_l - m_k} \leqslant -2 \times a_{m_k}$$

Since $m_l - m_k < m_l$ then by definition of m_k we get that $a_{m_k} \geqslant \max_{i<m_l-m_k}(a_i)$. By Lemma 2 we get that $|a_{m_l-m_k}| \leqslant 2 \times a_{m_k}$ which makes $|a_{m_l-m_k}| = 2 \times a_{m_k}$. In other word,

$$0 < a_{m_l} = -\max_{i+j=m_l}(a_i + a_j) \leqslant -(a_{m_t} + a_{m_l-m_k}) = a_{m_k}.$$

which contradicts to definition of m_l. Hence the sequence is bounded and we are done. □

Problem 3. Let $ABCD$ be a convex quadrilateral which has an inscribed circle centered at I. Let I_A, I_B, I_C, I_D be the incenters of triangles ABD, BCA, CBD, DAC respectively. The common external tangents of circumcircles of triangles AI_BI_D and CI_BI_D intersects at X. The common external tangents of circumcircles of triangles BI_AI_C and DI_AI_C intersects at Y. Prove that $\angle XIY = 90°$.

Solution 1. (Pitchayut Saengrungkongka)

By a well known lemma (or simple side-chasing), incircles of ΔABD, ΔCBD are tangent. So $BD \perp I_AI_C$. Furthermore since

$$\angle I_ABI_C = \frac{\angle B}{2} = \angle ABI$$

we get that $\angle I_CBD = \angle I_ABA = \angle I_ABD$ so $\{BI, BD\}$ are isogonal w.r.t. $\angle I_ABI_C$. But $BD \perp I_AI_C$ so the center O_B of $\odot(BI_AI_C)$ lies on BI. Similarly, the center O_D of $\odot(DI_AI_C)$ lies on DI.

Since $O_BO_D \perp I_AI_C \perp BD$ so $O_BO_D \parallel BD$. Hence

$$\frac{O_BY}{O_DY} = \frac{O_BB}{O_DD} = \frac{O_BI}{O_DI}$$

so IY externally bisects $\angle O_BIO_D \equiv \angle BID$. Similarly IX externally bisects $\angle AIC$.

Now it suffices to prove that IX internally bisects $\angle BID$ too, which is equivalent to $\{IA, IC\}$ are isogonal w.r.t. $\angle BID$. Finally, note by angle chasing that

$$\angle AID + \angle BIC = \left(180^\circ - \frac{\angle A}{2} - \frac{\angle D}{2}\right) + \left(180^\circ - \frac{\angle B}{2} - \frac{\angle D}{2}\right) = 180^\circ$$

implying the desired isogonality so we are done. □

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§3.9 Day 9 Solutions

Problem 1. Let $p \geqslant 2$ be prime number. Alice and Bob play a game which both players alternately make moves where Alice goes first. In each turn, player choose index i from the set $\{0, 1, ..., p-1\}$ that no one choose before then choose a_i from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. When all of the numbers $a_0, a_1, ..., a_{p-1}$ have been chosen, the game is ended and the value

$$M = a_0 + a_1 \cdot 10 + \cdots + a_{p-1} \cdot 10^{p-1}$$

is computed. Alice's goal is to make M divisible by p while Bob's goal is to prevent this. Prove that Alice has a winning strategy.

Solution 1. (Pitchayut Saengrungkongka)

We first eradicate the case $p = 2, 5$, which Alice just has to pick $a_0 = 0$ in the first move. Now let $k = \text{ord}_p(10)$, which clearly divides $p-1$. Let $l = \frac{p-1}{k}$. We split into two case.

Case 1 : k is even

Let $t = \frac{k}{2}$. It's easy to see that $10^t \equiv -1 \pmod p$. Now let Alice split $\{a_1, a_2, ..., a_{p-1}\}$ into $\frac{p-1}{2}$ pairs in form $\{a_n, a_{n+t}\}$ for some $n \in \mathbb{Z}^+$. It is straightforward to check that such pairing exists.

We define Alice's strategy as follows. Let she picks $a_0 = 0$ in her first move and makes other moves so that the two values in the same pair are forced to be equal. Clearly the sum contributed by pair $\{a_n, a_{n+t}\}$ is

$$a_n(10^{n+t} + 10^n) \equiv 0 \pmod p.$$

so Alice can force $p \mid M$ as desired.

Case 2 : k is odd

Since p is odd, l is even. Now let Alice split $\{a_0, a_1, ..., a_{p-1}\}$ into the following groups.

$$\begin{aligned} A_1 &= \{a_1, a_{k+1}, a_{2k+1}, ..., a_{(l-1)k+1}\} \\ A_2 &= \{a_2, a_{k+2}, a_{2k+2}, ..., a_{(l-1)k+2}\} \\ &\vdots \\ A_k &= \{a_k, a_{k+k}, a_{2k+k}, ..., a_{(l-1)k+k}\} \end{aligned}$$

Now define Alice's strategy as follows. Pick $a_0 = 0$ in her first move. Now whenever Bob picks the digit a in which group, Alice respond by insert another $9 - a$ in the same group, which is clearly possible since l is even. Clearly sum contributed by group A_n is

$$\equiv \frac{9 \cdot l}{2} \cdot 10^n \pmod p$$

Hence

$$M \equiv \frac{9l}{2}(10 + 10^2 + ... + 10^k) \equiv \frac{9l}{2} \cdot \frac{10(10^k - 1)}{10 - 1} \equiv 0 \pmod p$$

as desired. □

Remark 1. This problem works well with arbitrary base b. Except the case $p = 2$ and b is odd, which Bob can easily force a win.

Remark 2. The argument in Case 2 works well given only ℓ is even.

Problem 2. Let ω be the A-excircle of ΔABC, which touches lines BC, CA, AB at D, E, F respectively. Circumcircle of triangle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that circumcircle of triangle MPQ is tangent to ω.

Solution 1. (Pitchayut Saengrungkongka)

Let I_A be the center of ω, T is the second intersection of AD and ω. Let N be the midpoint of DT. Since $\angle AEI_A = \angle AFI_A = \angle ANI_A = 90°$, we get that $I_A, N \in \odot(AEF)$.

Now we claim that $MPQT$ is cyclic. Indeed, simply observe that

$$DP \cdot DQ = DA \cdot DN = DM \cdot DT.$$

Now let R be the point on BC such that RT is tangent to ω. Since R is the polar of AD w.r.t. ω, R must lie on EF by La Hire's Theorem. Hence

$$RP \cdot RQ = RE \cdot RF = RT^2$$

so RT also touches $\odot(MPQ)$ hence we are done. □

Solution 2. (Pitchayut Saengrungkongka)

Let I_A be the center of ω and let N be the midpoint of AI_A, which is the center of $\odot(AEF)$. Notice that MN is A-midline of ΔAI_AD so $MN \perp BC \implies MP = MQ$.

Claim. Circle $\Omega = \odot(M, MP)$ is orthogonal to ω.

Proof. Let P_1 be the reflection of P across M. Let $X = DP_1 \cap I_AP$. Notice that

$$DP_1 \parallel AP \perp PI_A \implies \angle PXP_1 = 90^\circ \implies X \in \Omega.$$

Moreover, $I_AX \cdot I_AP = I_AD^2$, implying the desired orthogonality. □

We can finish the solution with two different ways.

- Apply the converse of Casey's Theorem on degenerate circles M, P, Q and circle ω.
- Invert around Ω. Clearly it maps $\omega \to \omega$, $\odot(MPQ) \to PQ$, which obviously touches ω so we are done.

□

Problem 3. Let $n \geqslant 3$ be a positive integer. Call an n-tuple of real numbers $(x_1, x_2, ..., x_n)$ *shiny* if and only if for any permutation $(y_1, y_2, ..., y_n)$ of those number, we have

$$y_1y_2 + y_2y_3 + ... + y_{n-1}y_n \geqslant -1.$$

Determine the largest constant $K = K(n)$ such that the inequality

$$\sum_{1\leqslant i<j\leqslant n} x_ix_j \geqslant K$$

holds for all shiny n-tuple $(x_1, x_2, ..., x_n)$.

Solution 1. (Papon Lapate)

We claim that the answer is $\boxed{K(n) = \dfrac{1-n}{2}}$ for all $n \geqslant 3$.

First, let $(x_1, x_2, ..., x_n) = (a, a, ..., a, a, -\frac{1}{2a})$,we can see that this n-tuple is shiny and the summation is equal to $\binom{n-1}{2}a^2 + \frac{1-n}{2}$. Setting $a \to 0$ prove that $K \leqslant \frac{1-n}{2}$. Now it suffices to show that the inequality holds for $K = \frac{1-n}{2}$.

We will split into 2 cases.

Case 1. There are exactly $\frac{n}{2}$ positive numbers and $\frac{n}{2}$ negative numbers.

WLOG let $x_1, x_3, ...x_{n-1} < 0$ and $x_2, x_4, ...x_n > 0$. We also extend indices by $x_i = x_{i-n}$ for all $i > n$. Adding all cyclic variants of the assertion gives

$$y_1y_2 + y_2y_3 + ... + y_{n-1}y_n + y_ny_1 \geqslant -\frac{n}{n-1}.$$

Plugging in $(y_1, y_2, ..., y_n) = (x_1, x_{2k+2}, x_3, x_{2k+4}, ..., x_{n-1}, x_{2k+n})$ gives

$$x_1(x_{2k} + x_{2k+2}) + x_3(x_{2k+2}x_{2k+4}) + ... + x_{n-1}(x_{2k+n-2} + x_{2k+n}) \geqslant -\frac{n}{n-1}$$

Thus summing up from $k = 0, 2, 4, ..., n$ gives

$$\sum_{1\leqslant i,j\leqslant n, 2|i, 2\nmid j} x_ix_j \geqslant -\frac{n^2}{4(n-1)}.$$

Hence

$$\sum_{1\leqslant i<j\leqslant n} x_ix_j \geqslant \sum_{1\leqslant i,j\leqslant n, 2|i, 2\nmid j} x_ix_j \geqslant -\frac{n^2}{4(n-1)} \geqslant \frac{1-n}{2}$$

as desired.

Case 2. Number of positive and negative reals are different.

In this case, we can see that at least one of $y_1y_2, y_2y_3, ..., y_{n-1}y_n, y_ny_1$ must be non-negative. Hence

$$y_1y_2 + y_2y_3 + ... + y_{n-1}y_n + y_ny_1 \geqslant -1.$$

Summing all inequations of the previous form gives the desired result. □