InfinityDots MO3

Final Report

INFINITYDOTS

May 14, 2019

InfinityDots MO is an annual math contest started in 2017 simulating the International Mathematical Olympiad. InfinityDots MO 3 and JMO, held from 19 March to 8 April 2019, is the third iteration. It received a record number of participants and submissions. See more about InfinityDots MO at `https://www.infinitydots.org/mo.html`.

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P1. A set of two distinct coprime integers $\{x, y\}$ is said to be a *Pythagorean* if and only if $x^2 + y^2$ is an integer square. Given a Pythagorean, in each move, one can either

(i) change the sign of a number in the Pythagorean, or

(ii) add an integer k to both elements in the Pythagorean so that it is still a Pythagorean.

Show that starting from each Pythagorean, it is possible to reach any Pythagorean in a finite number of moves.

P2. Let $a_1, a_2, a_3, \ldots$ be a nonincreasing sequence of positive real numbers such that

$$a_n \geqslant a_{2n} + a_{2n+1} \text{ for all } n \geqslant 1.$$

Show that there exist infinitely many positive integers m such that

$$2m \cdot a_m > (4m - 3) \cdot a_{2m-1}.$$

P3. In a scalene triangle ABC, the incircle ω has center I and touches side BC at D. A circle Ω passes through B and C and intersects ω at two distinct points. The common tangents to ω and Ω intersect at T, and line AT intersects Ω at two distinct points K and L. Prove that either KI bisects $\angle AKD$ or LI bisects $\angle ALD$.

P4. An $n \times n$ table is written on a square piece of cardboard. Knuffle draws some diagonals in some of the n^2 cells, then uses a knife to cut along the marked diagonals. To Knuffle's surprise, the resulting piece of cardboard is still connected. Show that at least $2n-1$ cells were left uncut.

P5. Is there a nonempty finite set S of points on the plane that form at least $|S|^2$ harmonic quadrilaterals?
Note: a quadrilateral $ABCD$ is harmonic *if it is cyclic and $AB \cdot CD = BC \cdot DA$.*

P6. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that the set of functions $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$g(a)f(b) + g(b)f(a) \leqslant (a + f(a))(b + f(b)) \text{ for all } a, b \in \mathbb{R}$$

is finite but nonempty.

J1. Let $\mathbb{Z}_{>1}$ denote the set of all integers greater than 1. Is there a function $f : \mathbb{Z}_{>1} \to \mathbb{Z}_{>1}$ such that

$$f^{f(n)}(m) = m^n$$

for all integers m, n greater than 1?

Note: for any positive integer k, $f^k(n)$ denotes the result of f being applied k times to n.

J2. Find all pairs (a, b) of positive integers such that $(a+1)^{b-1} + (a-1)^{b+1} = 2a^b$.

J3. There is a calculator with a display and two buttons: $-1/x$ and $x+1$. The display is capable of displaying precisely any arbitrary rational numbers. The buttons, when pressed, will change the value x displayed to the value of the term on the button. (The $-1/x$ button cannot be pressed when $x = 0$.)

At first, the calculator displays 0. You accidentally drop the calculator on the floor, resulting in the two buttons being pressed a total of N times in some order. Prove that you can press the buttons at most $3N$ times to get the display to show 0 again.

Note: partial credit will be given for showing a bound of cN for a constant $c > 3$.

J4. A sequence $a_1, a_2, \ldots$ of positive integers satisfies

$$a_n = \sqrt{(n+1)a_{n-1} + 1} \quad \text{for all } n \geqslant 2.$$

What are the possible values of a_1?

J5. A positive integer $n > 2$ is chosen, and each of the numbers $1, 2, \ldots, n$ is colored red or blue. Show that it is possible to color each subset of $\{1, 2, \ldots, n\}$ either red or blue so that each red number lies in more red subsets than blue subsets, and each blue number lies in more blue subsets than red subsets.

J6. Determine all positive reals r such that, for any triangle ABC, we can choose points D, E, F trisecting the perimeter of the triangle into three equal-length sections so that the area of $\triangle DEF$ is exactly r times that of $\triangle ABC$.

Solutions

P1. A set of two distinct coprime integers $\{x, y\}$ is said to be a *Pythagorean* if and only if $x^2 + y^2$ is an integer square. Given a Pythagorean, in each move, one can either

(i) change the sign of a number in the Pythagorean, or

(ii) add an integer k to both elements in the Pythagorean so that it is still a Pythagorean.

Show that starting from each Pythagorean, it is possible to reach any Pythagorean in a finite number of moves.

Proposed by talkon

Solution (1: introducing z)**.** We note that if $x^2 + y^2 = z^2$, then $k_0 = 2z - 2x - 2y$ satisfy

$$(x + k_0)^2 + (y + k_0)^2 = (z + k_0)^2,$$

therefore the move $\{x, y\} \to \{x + k_0, y + k_0\}$ is valid. Furthermore, if x, y, z are positive then $2z > x + y > \sqrt{x^2 + y^2} = z$ so

$$-z < z + k_0 = 3z - 2x - 2y < z.$$

As long as $x, y \neq 0$, we can make x, y, z positive and then use $\{x, y\} \to \{x + k_0, y + k_0\}$ to reduce the value of z. As z cannot decrease indefinitely, we must reach a state where one of x, y is 0, and that is only possible when $\{x, y\} = \{1, 0\}$. Therefore, starting from any Pythagorean $\{x, y\}$, it is possible to reach $\{1, 0\}$, from which we can apply the moves in reverse to get to any Pythagorean $\{x', y'\}$. ■

Solution (2: Euclid's formula)**.** Note that moves of type (i) allows us to ignore signs of numbers in a Pythagorean. Define $[u, v] = \{u^2 - v^2, 2uv\}$. Euclid's formula says that each Pythagorean $\{x, y\}$ is equal to $[u, v]$ for some coprime nonnegative u, v with u odd and v even, modulo signs.

The identity

$$(a^2 - b^2) - \big((a - 2b)^2 - b^2\big) = 2ab - \big(-2(a - 2b)b\big)$$

implies that the moves $[u, v] \to [|u - 2v|, v]$ and $[u, v] \to [u, |v - 2u|]$ are valid. Applying these moves repeatedly, reducing u when $u > v$ and v when $v > u$, we can reduce each Pythagorean $[u, v]$ to $[1, 0]$, then we can apply the moves in reverse to go from $[1, 0]$ to any Pythagorean $[u', v']$. ■

P2. Let $a_1, a_2, a_3, \ldots$ be a nonincreasing sequence of positive real numbers such that

$$a_n \geqslant a_{2n} + a_{2n+1} \text{ for all } n \geqslant 1.$$

Show that there exist infinitely many positive integers m such that

$$2m \cdot a_m > (4m-3) \cdot a_{2m-1}.$$

Proposed by talkon

Solution (1: telescope)**.** Assume for the sake of contradiction that $2m \cdot a_m \leqslant (4m-3) \cdot a_{2m-1}$ for all $m > N$. Replacing m by $m+1$ gives

$$a_{2m+1} \geqslant \frac{2(m+1)}{4m+1} a_{m+1}.$$

For each $n > N$ and $k \geqslant 0$, we can repeatedly apply the above for $m = n, 2n, \ldots, 2^{k-1}n$, getting

$$\begin{aligned}
a_{2^k n+1} &\geqslant \frac{2\cancel{(2^{k-1}n+1)}}{(2^{k+1}n+1)} \cdot \frac{2\cancel{(2^{k-2}n+1)}}{(2^k n+1)} \cdot \cdots \cdot \frac{2(2n+1)}{\cancel{8n+1}} \cdot \frac{2(n+1)}{\cancel{4n+1}} \cdot a_{n+1} \\
&= \frac{2^k(n+1)(2n+1)}{(2^k n+1)(2^{k+1}n+1)} \cdot a_{n+1} \\
&= \frac{(n+1)(2n+1)}{m} \cdot \left(\frac{1}{2^k n+1} - \frac{1}{2^{k+1}n+1} \right) \cdot a_{n+1}
\end{aligned}$$

Therefore, for all $n > N$,

$$\begin{aligned}
a_n &\geqslant a_{2n+1} + a_{2n} \\
&\geqslant a_{2n+1} + a_{4n+1} + a_{4n} \\
&\geqslant \cdots \\
&\geqslant \sum_{k=1}^{\infty} a_{2^k n+1} \\
&\geqslant a_{n+1} \cdot \frac{(n+1)(2n+1)}{n} \sum_{k=1}^{\infty} \left(\frac{1}{2^k n+1} - \frac{1}{2^{k+1}n+1} \right) \\
&= a_{n+1} \cdot \frac{n+1}{n}.
\end{aligned}$$

Hence

$$a_n \geqslant a_{n+1} \cdot \frac{n+1}{n} \geqslant \cdots \geqslant a_{2n-1} \cdot \frac{2n-1}{\cancel{2n-2}} \cdot \frac{\cancel{2n-2}}{\cancel{2n-3}} \cdot \cdots \cdot \frac{\cancel{n+1}}{n} > a_{2n-1} \cdot \frac{4n-3}{2n},$$

which is the desired contradiction. ■

Solution (2: iterative bound)**.** Suppose for the sake of contradiction that $a_n \leqslant \frac{4m-3}{2m} \cdot a_{2m-1}$ for all big enough n, say, $n > N$.

We will prove, by induction on k, that

$$\frac{a_n}{a_{n-1}} \leqslant \frac{n - 1 + \frac{1}{2^k}}{n}$$

for all integers $n > N$ and nonnegative integer k.

- **Base case:** $k = 0$. Since $\{a_n\}$ is decreasing, $\dfrac{a_n}{a_{n-1}} \leqslant 1 = \dfrac{n-1+\frac{1}{2^0}}{n}$ for all $n > N$.
- **Inductive step:** Assume $\dfrac{a_n}{a_{n-1}} \leqslant \dfrac{n-1+\frac{1}{2^k}}{n}$ for all $n > N$.

  In particular, replacing n with $2n-1 > N$ gives

  $$\frac{a_{2n-1}}{a_{2n-2}} \leqslant \frac{2n-2+\frac{1}{2^k}}{2n-1}$$

  which, since $a_{n-1} = a_{2n-1} + a_{2n}$, implies

  $$a_{2n-1} \leqslant \frac{2n-2+\frac{1}{2^k}}{4n-3+\frac{1}{2^k}} \cdot a_{n-1},$$

  therefore

  $$a_n \leqslant \frac{4n-3}{2n} \cdot \frac{2n-2+\frac{1}{2^k}}{4n-3+\frac{1}{2^k}} \cdot a_{n-1} \leqslant \frac{n-1+\frac{1}{2^{k+1}}}{n} \cdot a_{n-1},$$

  completing the inductive step.

Clearly it follows that for all $n > N$, $\frac{a_n}{a_{n-1}} \leqslant \frac{n-1}{n}$. Hence

$$\frac{a_{2n-1}}{a_n} = \frac{a_{2n-1}}{a_{2n-2}} \cdot \frac{a_{2n-1}}{a_{2n-2}} \cdot \dots \cdot \frac{a_{n+1}}{a_n} \leqslant \frac{2n-2}{2n-1} \cdot \frac{2n-3}{2n-2} \cdot \dots \cdot \frac{n}{n+1} = \frac{n}{2n-1}$$

so $2n \cdot a_n \geqslant (4n-2)a_{2n-1} > (4n-3)a_{2n-1}$, which is the desired contradiction. ■

Comments. As seen in Solution 1, the decreasing condition is superfluous and not actually needed. If the condition is removed, it is possible to modify Solution 2 by adding another iterative bound: that $\dfrac{a_n}{a_{n-1}} < 1 + \dfrac{1}{2^k}$ for all integers $n > N$ and nonnegative integer k.

- **Base case:** $k = 0$. We have

  $$a_{n-1} = a_{2n-2} + a_{2n-1} > a_{2n-1} \geqslant \frac{2n}{4n-3} a_n > \frac{1}{2} a_n$$

  for all $n > N$, so $\dfrac{a_n}{a_{n-1}} < 2 = 1 + \dfrac{1}{2^0}$.
- **Inductive step:** Assume $\dfrac{a_n}{a_{n-1}} < 1 + \dfrac{1}{2^k}$ for all $n > N$.

  Replacing n with $2n-1 > N$ gives $\dfrac{a_{2n-1}}{a_{2n-2}} < 1 + \dfrac{1}{2^k}$, which, since $a_{n-1} = a_{2n-1} + a_{2n}$, implies

  $$a_{2n-1} < \frac{2^k+1}{2^{k+1}+1} \cdot a_{n-1},$$

  therefore

  $$a_n < \frac{4n-3}{2n} \cdot \frac{2^k+1}{2^{k+1}+1} \cdot a_{n-1} < \frac{2^{k+1}+2}{2^{k+1}+1} \cdot a_{n-1} < \left(1 + \frac{1}{2^{k+1}}\right) a_{n-1},$$

  completing the inductive step. □

P3. In a scalene triangle ABC, the incircle ω has center I and touches side BC at D. A circle Ω passes through B and C and intersects ω at two distinct points. The common tangents to ω and Ω intersect at T, and line AT intersects Ω at two distinct points K and L. Prove that either KI bisects $\angle AKD$ or LI bisects $\angle ALD$.

Proposed by talkon and ThE-dArK-lOrD

Solution (1: theorem medley)**.** We divide the proof into two steps.

STEP 1. We show that either KI bisects $\angle BKC$ or LI bisects $\angle BLC$.

Construct the circle Γ tangent to AB, AC and Ω internally; let K' be the tangency point of Γ and Ω. By MONGE'S THEOREM, K' is on line AT, so $K' \in \{K, L\}$. By PROTASSOV'S THEOREM[1], $K'I$ bisects $\angle BK'C$, so either KI bisects $\angle BKC$ or LI bisects $\angle BLC$. Without loss of generality, assume the former.

STEP 2. If KI bisects $\angle BKC$ then it also bisects $\angle AKD$.

Let the tangents from K to ω meet ω at X, Y. The DUAL OF DESARGUE'S INVOLUTION THEOREM on the degenerate quadrilateral $ABDC$ with inscribed circle ω says that there is an involution swapping $\{X, Y\}, \{B, C\}, \{A, D\}$. As KI bisects $\angle XKY$ and $\angle BKC$, this involution is the reflection over $\overleftrightarrow{KI}$, so KI also bisects $\angle AKD$. ∎

Solution (2: alternatives)**.** We provide alternative ways for both of the steps above.

STEP 1. We show that either KI bisects $\angle BKC$ or LI bisects $\angle BLC$.

Define coordinates so that the midpoint of BC is $(0,0)$, B, C lie on the x-axis, ω is above BC, and the radius r of ω is 1. Let $F, J; M, N$ be the bottommost, topmost points of $\omega; \Omega$ respectively. Let K' be the second intersection of NI and Ω. Clearly $K'I$ bisects $\angle BK'C$, so it suffices to show that K' lies on $\overleftrightarrow{AT}$.

Let $B = (-b, 0), C = (b, 0), N = (0, n), F = (f, 0)$. By power-of-point, $M = (0, -b^2/n)$. Since $r = 1$, $I = (f, 1)$ and $J = (f, 2)$.

Finding A: the homothety sending ω to the A-excircle ω_A is centered at A and sends J to $L' = (0, f)$. For the moment, define x, y, z as in the Ravi substitution, that is, x, y, z are the lengths of tangents from A, B, C to ω respectively. Also let r_a be the radius of ω_A. We know that $1 = r = \frac{\triangle}{s} = \sqrt{\frac{xyz}{x+y+z}}$, so the scaling factor of the homothety is $\frac{r_a}{r} = \frac{x+y+z}{x} = yz = b^2 - f^2$ as $y = b + f$ and $z = b - f$. Therefore, $(A - L') = (b^2 - f^2)(A - J)$, that is,

$$A = \frac{1}{b^2 - f^2 - 1}\left((b^2 - f^2) \cdot J - L'\right) = \frac{1}{b^2 - f^2 - 1} \cdot \left(f(b^2 - f^2 + 1), 2(b^2 - f^2)\right).$$

Finding T: consider another homothety, this time sending ω to Ω. This is centered at T and sends J to N. The radius R of Ω is $\frac{1}{2}\left(n + \frac{b^2}{n}\right)$, which is also the scaling factor. Therefore, $(T - N) = \frac{1}{2}\left(n + \frac{b^2}{n}\right)(T - J)$, that is,

$$T = \frac{2}{n + \frac{b^2}{n} - 2}\left(\frac{n + \frac{b^2}{n}}{2} \cdot J - N\right) = \frac{1}{b^2 + n^2 - 2n}\left(f(b^2 + n^2), 2b^2\right).$$

[1] PROTASSOV'S THEOREM: Given $\triangle ABC$ with incenter I. A circle Ω passes through B, C. Suppose that a circle Γ is tangent to AB, AC, and to Ω internally at P. Then PI bisects $\angle BPC$.

Finding K': we simply need to project M onto $\overleftrightarrow{NI}$. The equation for $\overleftrightarrow{NI}$ is $y = \frac{n-1}{f} \cdot x + n$, so the equation of the line through M perpendicular to $\overleftrightarrow{NI}$ is $y = \frac{f}{1-n} \cdot x - \frac{b^2}{n}$. Solving the system of equation gives

$$K' = \frac{1}{n\left((n-1)^2 + f^2\right)}\left((n-1)f(b^2+n^2), n^2f^2 - b^2(n-1)^2\right).$$

Finale: Using the shoelace formula and multiplying by appropriate constants, we want to show that

$$\det\begin{vmatrix} b^2 - f^2 - 1 & f(b^2 - f^2 + 1) & 2(b^2 - f^2) \\ b^2 + n^2 - 2n & f(b^2+n^2) & 2b^2 \\ n\left((n-1)^2 + f^2\right) & (n-1)f(b^2+n^2) & n^2f^2 - b^2(n-1)^2 \end{vmatrix} = 0.$$

This can be verified by direct computation or by using row/column operations to ease the computation. The following is one way to do it. Let c_1, c_2, c_3 denote the columns. Modify the columns by $c_2 \to \frac{c_2}{f}$ (it's clear that if $f = 0$ the determinant vanishes) then $c_3 \to \frac{c_3 - c_1 - c_2}{n - n^2}$ (same reasoning) gives

$$\begin{vmatrix} b^2 - f^2 - 1 & b^2 - f^2 + 1 & 0 \\ b^2 + n^2 - 2n & b^2 + n^2 & 2 \\ n\left((n-1)^2 + f^2\right) & (n-1)(b^2+n^2) & b^2 - f^2 - 1 + 2n \end{vmatrix}.$$

Taking $c_1 \to -c_1 + c_2$ gives

$$\begin{vmatrix} 2 & b^2 - f^2 + 1 & 0 \\ 2n & b^2 + n^2 & 2 \\ b^2n - b^2 - f^2n + n^2 - n & (n-1)(b^2+n^2) & b^2 - f^2 - 1 + 2n \end{vmatrix}.$$

If the determinant were to be 0, $(b^2 - f^2 + 1)c_1 - 2c_2$ needs to be a multiple of c_3, say, mc_3. The second row forces m to be $n(b^2 - f^2 + 1) - (b^2 + n^2) = b^2n - b^2 - f^2n - n^2 + n$. Finally, we check that

$$\begin{aligned}
&(b^2 - f^2 + 1)(b^2n - b^2 - f^2n + n^2 - n) - (b^2n - b^2 - f^2n - n^2 + n)(b^2 - f^2 - 1 + 2n) \\
&= 2(n^2 - n)(b^2 - f^2 + 1) - 2(n-1)(b^2n - b^2 - f^2n - n^2 + n) \\
&= 2(n-1)(nb^2 - f^2n + n - b^2n + b^2 + f^2n + n^2 - n) \\
&= 2(n-1)(b^2 + n^2)
\end{aligned}$$

as expected.

STEP 2. If KI bisects $\angle BKC$ then it also bisects $\angle AKD$.

Let M and N denote the midpoints of arc BC containing and not containing K respectively. Let MK intersects BC at H, MR intersects Ω again at P, and AT intersects Ω at $Q \neq K$. Let PI intersects Ω again at P'. Note that K, I, N collinear and that H, B, C collinear in this order.

We have $\angle IKH = 90° = \angle IRH$, so $IRKH$ is cyclic. Let κ denote its circumcircle. Also, we have $\angle KIR = \angle KNM = \angle KPM \implies P \in (KRI) = \kappa$. So, $\angle RHI = \angle RPI = \angle MPP'$.

Also, since $HC \perp MN$ and $NK \perp MH$, we get $\angle NHR = \angle KNM$. Hence, we get $\angle KHI = \angle KNP' = 180^\circ - \angle MP' \implies HI \parallel MP'$.

Applying PASCAL'S THEOREM on six points Q, K, N, M, P, P' gives us Q, O, P' collinear, so P' is the antipode of Q in Ω.

As $IH \parallel MP'$, $IH \perp MQ$. Homothethy gives us $IH \perp RS$ where S is the intersection of segment AT and ω. Not hard to get that S is the intersection point of second tangent from H to ω. Hence $\angle HSI = 90^\circ \implies S \in \kappa$. Since $IR = IS$, we are done. ■

Comments. 1. The problem was proposed without knowledge of PROTASSOV'S THEOREM; the original version of the problem was:

Let $\square ABCD$ be a convex quadrilateral such that the incircle of $\triangle ABC$ is tangent to the circumcircle of $\triangle ADC$ at a point on line BD. Show that the incenter of $\triangle ABC$ lies on the angle bisector of $\angle ADC$.

but this turned out to known as a variation on PROTASSOV'S THEOREM, so we opted for the generalization. Some possible proofs of this include:

- Inversion around the incircle of $\triangle ABC$ and possibly another $\sqrt{bc}$-inversion
- Desargue's Involution Theorem on $ABCD$ and ω
- The coordinate bash presented above

2. If T is replaced by the insimilicenter instead, and K is chosen on the arc BC not containing D, it still holds true that KI bisects $\angle BKC$ and $\angle AKB$. We believe that this can be shown using a similar method the solutions presented.

3. Finally, there is another generalization which seems to be true, but we were unable to prove:

Let $\square ABCD$ be a convex quadrilateral such that the incircle of $\triangle ABC$ is tangent to the circumcircle of $\triangle ADC$ at a point K. Then any two of the following implies the other:

- *K lies on BD*
- *The tangency point T of the incircle of $\triangle ADC$ and AC lies on BD*
- *The incircle of $\triangle ADC$ is tangent to the circumcircle of $\triangle ABC$*

P4. An $n \times n$ table is written on a square piece of cardboard. Knuffle draws some diagonals in some of the n^2 cells, then uses a knife to cut along the marked diagonals. To Knuffle's surprise, the resulting piece of cardboard is still connected. Show that at least $2n-1$ cells were left uncut.

Proposed by ThE-dArK-lOrD

Solution (1: graph on vertices)**.** If a cell has two cut diagonals, pretend only one is cut–the cardboard would still be connected. Construct a graph G with $(n-1)^2+1$ vertices: the $(n-1)^2$ interior vertices of the square plus an extra vertex for the whole boundary. Draw edges joining two vertices iff the diagonal connecting the two are cut. As the cardboard is still connected, G contain no cycles. Therefore the number of cuts made, i.e. the number of edges in G, is $\leqslant (n-1)^2$. Hence at least $n^2-(n-1)^2 = 2n-1$ cells were left uncut. ■

Solution (2: graph on edges)**.** Consider a graph with the $2n(n+1)$ edge segments of the square as vertices. For each cut square, draw two edges joining pairs of edge segments on the same side. For each uncut square, draw three edges joining the four edge segments. If there are k uncut squares, G will have $2n^2+k$ edges. However, G must be connected, so

$$k \geqslant 2n(n+1) - 1 - 2n^2 = 2n-1. \quad ■$$

Solution (3: graph on regions)**.** Consider a graph where the vertices are the parts of the table separated by gridlines and drawn diagonals, and there is an edge connecting two vertices iff the two parts share a section of a gridline.

Observe that each cell with a drawn diagonal increases $|V|$ by at least 1, so if t is the number of cut cells then $|V| \geqslant n^2+t$. Also, as there are $2n(n-1)$ sections of gridline inside the table, $|E|$ is exactly $2n(n-1)$. Finally, as the piece of cardboard is still connected after diagonal cuts, G is connected, so

$$n^2+t = |V| \leqslant |E|+1 = 2n^2-2n+1,$$

hence $t \leqslant n^2-2n+1$ ■

Solution (4: boundary components)**.** We claim that if exactly one diagonal is cut in all n^2 cells of the table, then the table is cut into at least $2n$ pieces. To show that, consider the graph where the $2n(n+1)$ gridline sections are the vertices, and two vertices are connected by an edge iff they lie on the same half-cell. In this graph, vertex has degree exactly two except the $4n$ boundary vertices which have degree one. Each component of this graph can only contain two boundary vertices, so there are at least $4n/2 = 2n$ components.

From our claim, the problem is straightforward: by allowing cells to have more than one cut diagonal, the number of components cannot be reduced, and for each uncut cell, the number of components can decrease by at most one, so in order to have a single component, at least $2n-1$ cells must be uncut. ■

Comments. The problem also holds for rectangular tables, where for an $m \times n$ table, at least $m+n-1$ cells must be left uncut. All solutions presented above can be easily modified for the rectangular case.

P5. Is there a nonempty finite set S of points on the plane that form at least $|S|^2$ harmonic quadrilaterals?

Note: a quadrilateral $ABCD$ is harmonic *if it is cyclic and $AB \cdot CD = BC \cdot DA$.*

Proposed by talkon

Answer. Yes.

Solution. Define points $P_i = (i, 0)$ for $i = 0, \dots, N-1$. We claim that for sufficiently large N, the set of N points obtained by inverting $\{P_0, \dots, P_{N-1}\}$ around any point not on x-axis satisfies the condition. Since the cross-ratio of any four points is invariant under inversion, it suffices to show that there are at least N^2 *good* tuples (a, b, c, d) with $0 \leqslant a < b < c < d \leqslant N$ such that $(P_a, P_c; P_b, P_d) = -1$.

Call a good tuple (a, b, c, d) *primitive* if $a = 0$ and $\gcd(b, c, d) = 1$. Note that for each good tuple (p, q, r, s) there is a unique primitive tuple (a, b, c, d) for which there exists integers $t \geqslant 1, k \geqslant 0$ satisfying $(p, q, r, s) = (ta + k, tb + k, tc + k, td + k)$. Also, for each primitive tuple (a, b, c, d), the tuple $(ta + k, tb + k, tc + k, td + k)$ is good for any positive integers $t \geqslant 1, k \geqslant 0$ that $td + k < N$, and there are at least

$$\begin{aligned}&(N-d) + (N-2d) + \dots + (N - \lfloor N/d \rfloor d)\\ &\geqslant N\lfloor N/d \rfloor - d(1 + 2 + \dots + \lfloor N/d \rfloor)\\ &\geqslant N(N/d - 1) - d \cdot \frac{\frac{N}{d}\left(\frac{N}{d} + 1\right)}{2}\\ &\geqslant \frac{N^2}{2d} - 2N\end{aligned}$$

choices of (t, k).

Now, observe that the tuple $(0, (2m-n)m, (2m-n)n, mn)$ with $m < n$ positive integers, n odd, and $\gcd(m, n) = 1$ is primitive. Therefore, for each odd integer $n > 1$, there are $\varphi(n)$ choices of m. As each (m, n) gives $\frac{N^2}{2mn} - 2N > \frac{N^2}{2n^2} - 2N$ choices of (t, k), there are at least

$$\frac{N^2}{2} \cdot \frac{\varphi(n)}{n^2} - 2N\varphi(n) > \frac{N^2}{2} \cdot \frac{\varphi(n)}{n^2} - 2Nn$$

tuples formed from a fixed value of n.

As $\displaystyle\sum_{n \text{ odd}} \frac{\varphi(n)}{n^2} > \sum_{p \text{ odd prime}} \frac{\varphi(p)}{p^2} = \sum_{p \text{ odd prime}} \frac{p-1}{p^2}$ which diverges, $\displaystyle\sum_{n \text{ odd}} \frac{\varphi(n)}{n^2}$ also diverges.

Pick a cutoff K such that $\displaystyle\sum_{\substack{n \text{ odd} \\ n<K}} \frac{\varphi(n)}{n^2} > 3$. It follows that, for $N > K^2$, there are at least

$$\left(\frac{N^2}{2} \sum_{\substack{n \text{ odd} \\ n<K}} \frac{\varphi(n)}{n^2}\right) - 2N \sum_{\substack{n \text{ odd} \\ n<K}} n > \frac{3N^2}{2} - 2N\frac{K^2}{4} > N^2$$

good tuples. ■

Comments. In fact, for each pair of positive integers $m < n$ with n odd and $\gcd(m, n) = 1$, $(0, mn - (2m-n)m, mn - (2m-n)n, mn)$ is also primitive so the number of good tuples is undercounted by at least a factor of 2.

P6. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that the set of functions $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$g(a)f(b) + g(b)f(a) \leqslant (a + f(a))(b + f(b)) \text{ for all } a, b \in \mathbb{R}$$

is finite but nonempty.

Proposed by TLP.39

Answer. All continuous functions f that satisfy either

$$\inf_{x>0} \frac{f(x)}{x} = \sup_{x<0} \frac{f(x)}{x} \neq 0, \quad \text{or} \quad \inf_{x<0} \frac{f(x)}{x} = \sup_{x>0} \frac{f(x)}{x} \neq 0.$$

Solution. Rewrite the inequality as follows:

$$f(a)\left(g(b) - \frac{f(b)}{2} - b\right) + f(b)\left(g(a) - \frac{f(a)}{2} - a\right) \leqslant ab \quad \text{for all } a, b \in \mathbb{R}.$$

Defining $h :\ \mathbb{R} \to \mathbb{R}$ by $h(x) = g(x) - \dfrac{f(x)}{2} - x$ for all $x \in \mathbb{R}$, we get that the set

$$H_f := \{h : \mathbb{R} \to \mathbb{R} \mid h(a)f(b) + h(b)f(a) \leqslant ab \text{ for all } a, b \in \mathbb{R}\}$$

must also be finite and nonempty.

Lemma 1. $|H_f| < 2$

Proof. Suppose there exist two distinct elements h_1 and h_2 of H_f. For any $\lambda \in (0, 1)$, define $h_3(x) = \lambda h_1(x) + (1 - \lambda)h_2(x)$. We have

$$\begin{aligned} h_3(a)f(b) + h_3(b)f(a) &= \lambda\left(h_1(a)f(b) + h_1(b)f(a)\right) + (1 - \lambda)\left(h_2(a)f(b) + h_2(b)f(a)\right) \\ &\leqslant \lambda ab + (1 - \lambda)ab = ab. \end{aligned}$$

So $h_3(x)$ must be an element of H_f for all $\lambda \in (0, 1)$, which contradicts the finiteness of H_f. □

By the above lemma, $|H_f| = 1$ and so let $H_f = \{h\}$.

Lemma 2. *The range of f must contain both positive and negative numbers.*

Proof. WLOG, we may assume for the sake of contradiction that $f(x) \geq 0$ for all $x \in \mathbb{R}$. Consider the function $h_2(x) = h(x) - \alpha$ where α be any positive real constant. We have

$$\begin{aligned} h_2(a)f(b) + h_2(b)f(a) &= h(a)f(b) + h(b)f(a) - \alpha(f(b) + f(a)) \\ &\leqslant h(a)f(b) + h(b)f(a) \leqslant ab \end{aligned}$$

Therefore, h_2 is also an element of H_f, which contradicts the uniqueness of h. □

LEMMA 3. *There exists exactly one value of k such that either $f(x) \leqslant kx$ for all $x \in \mathbb{R}$ or $f(x) \geqslant kx$ for all $x \in \mathbb{R}$. Moreover, for that value of k, we have $k \neq 0$.*

Proof. From LEMMA 2, for any (not necessary distinct) real numbers a, b, c, d such that $f(a), f(b) > 0$ and $f(c), f(d) < 0$, we obtain the following inequalities

$$\frac{h(a)}{f(a)} + \frac{h(b)}{f(b)} \leqslant \frac{a}{f(a)} \cdot \frac{b}{f(b)}$$
$$\frac{h(c)}{f(c)} + \frac{h(d)}{f(d)} \leqslant \frac{c}{f(c)} \cdot \frac{d}{f(d)}$$
$$\frac{h(a)}{f(a)} + \frac{h(c)}{f(c)} \geqslant \frac{a}{f(a)} \cdot \frac{c}{f(c)}$$
$$\frac{h(b)}{f(b)} + \frac{h(d)}{f(d)} \geqslant \frac{b}{f(b)} \cdot \frac{d}{f(d)}$$

Combining the above inequalities gives us

$$\left(\frac{a}{f(a)} - \frac{d}{f(d)}\right)\left(\frac{b}{f(b)} - \frac{c}{f(c)}\right) \geqslant 0.$$

Hence, there must exist a constant t such that either

1. $\dfrac{a}{f(a)} \geqslant t \geqslant \dfrac{c}{f(c)}$ for all a and c that $f(a) > 0$ and $f(c) < 0$, or
2. $\dfrac{a}{f(a)} \leqslant t \leqslant \dfrac{c}{f(c)}$ for all a and c that $f(a) > 0$ and $f(c) < 0$.

In both cases, either $t = 0$ or there exists constant $k = 1/t \neq 0$ such that $f(x) \geqslant kx$ for all $x \in \mathbb{R}$ that $f(x) \neq 0$ or $f(x) \leqslant kx$ for all $x \in \mathbb{R}$ that $f(x) \neq 0$.

However, if $t = 0$, then both cases imply that either

1. $a \geqslant 0$ for all a such that $f(a) \neq 0$, or
2. $a \leqslant 0$ for all a such that $f(a) \neq 0$.

In other words, either $f(a) = 0$ for all $a > 0$ or $f(a) = 0$ for all $a < 0$. Thus, by continuity of f, either $f(a) = 0$ for all $a \geqslant 0$ or $f(a) = 0$ for all $a \leqslant 0$.

Suppose $f(a) = 0$ for all $a \leqslant 0$. (The other case can be done analogously.) Since there exist a positive real number and a negative real number in the range of f, there must also exist $a, b > 0$ such that $f(a) > 0 > f(b)$. Hence for any $c < 0$, we have

$$f(a)g(c) \leqslant ac < 0 \implies g(c) < 0$$

and

$$f(b)g(c) \leqslant bc < 0 \implies g(c) > 0$$

which lead to a contradiction. So, we conclude that the case when $t = 0$ can't happen.

Now, without loss of generality, we assume that $f(x) \leqslant kx$ for all $x \in \mathbb{R}$ that $f(x) \neq 0$ and also $k > 0$ (The other cases can be done analogously.)

If there is a real number q such that $f(q) > kq$, we get that $f(q) = 0$ and thus $0 = f(q) > kq \implies q < 0$. By continuity of f, we obtain that $f(x) = 0$ for all $x \leqslant 0$, and so a contradiction follows by the same argument as in the case $t = 0$. So, we conclude that $f(x) \leqslant kx$ for all $x \in \mathbb{R}$. Furthermore, observe that the function $h(x) = -\dfrac{f(x)}{2k^2} + \dfrac{x}{k}$ for all $x \in \mathbb{R}$ satisfies the condition because $\left(\dfrac{f(a)}{k} - a\right)\left(\dfrac{f(b)}{k} - b\right) \geqslant 0$ for all $a, b \in \mathbb{R}$.

If there is another constant $k' \neq k$ such that $f(x) \leqslant k'x$ for all $x \in \mathbb{R}$, then we get that $f(x)$ is not linear and the function $h(x) = -\dfrac{f(x)}{2k'^2} + \dfrac{x}{k'}$ for all $x \in \mathbb{R}$ also satisfies the condition. This contradicts Lemma 1. Thus, we've proved the existence of k as stated at the beginning.

The fact that $k \neq 0$ follows immediately as a consequence of Lemma 2. □

For the rest of the solution, we'll show that the condition in Lemma 3 is enough to guarantee the uniqueness of element of H_f.

By Lemma 3, there exist sequences of positive real numbers $\{r_n\}$ and of negative real numbers $\{s_n\}$ such that $\lim_{n\to\infty} \dfrac{f(r_n)}{r_n} = \lim_{n\to\infty} \dfrac{f(s_n)}{s_n} = k$. Now, for any fixed r_i and s_j, note that $f(r_i) > 0$ and $f(s_j) < 0$. We, then, have

$$\begin{aligned}
h(r_i)f(s_j) + h(s_j)f(r_i) \leqslant r_i s_j &\implies \frac{h(r_i)}{f(r_i)} + \frac{h(s_j)}{f(s_j)} \geqslant \frac{r_i}{f(r_i)} \cdot \frac{s_j}{f(s_j)} \\
2h(r_i)f(r_i) \leqslant r_i^2 &\implies \frac{h(r_i)}{f(r_i)} \leqslant \frac{r_i^2}{2f(r_i)^2} \\
2h(s_j)f(s_j) \leqslant s_j^2 &\implies \frac{h(s_j)}{f(s_j)} \leqslant \frac{s_j^2}{2f(s_j)^2}.
\end{aligned}$$

Combining all the inequalities, we obtain

$$\frac{r_i^2}{2f(r_i)^2} \geqslant \frac{h(r_i)}{f(r_i)} \geqslant \frac{r_i}{f(r_i)} \cdot \frac{s_j}{f(s_j)} - \frac{s_j^2}{2f(s_j)^2}$$

and

$$\frac{s_j^2}{2f(s_j)^2} \geqslant \frac{h(s_j)}{f(s_j)} \geqslant \frac{r_i}{f(r_i)} \cdot \frac{s_j}{f(s_j)} - \frac{r_i^2}{2f(r_i)^2}.$$

On the other hand, we have

$$\begin{aligned}
h(r_i)f(y) + h(y)f(r_i) \leqslant r_i \cdot y &\implies h(y) \leqslant \frac{r_i}{f(r_i)} \cdot y - \frac{h(r_i)}{f(r_i)} \cdot f(y) \\
h(s_j)f(y) + h(y)f(s_j) \leqslant s_j \cdot y &\implies h(y) \geqslant \frac{s_j}{f(s_j)} \cdot y - \frac{h(s_j)}{f(s_j)} \cdot f(y).
\end{aligned}$$

Hence, choosing $i, j \to \infty$, we get $h(y) = -\dfrac{f(y)}{2k^2} + \dfrac{y}{k}$ for all $y \in \mathbb{R}$. Note that such function satisfies the condition as shown as part of the proof of Lemma 3. ■

J1. Let $\mathbb{Z}_{>1}$ denote the set of all integers greater than 1. Is there a function $f : \mathbb{Z}_{>1} \to \mathbb{Z}_{>1}$ such that

$$f^{f(n)}(m) = m^n$$

for all integers m, n greater than 1?
Note: for any positive integer k, $f^k(n)$ denotes the result of f being applied k times to n.

Proposed by jeneva

Answer. No.

Solution. Suppose there exists such an f. Clearly, f is injective. Fix an $m \in \mathbb{Z}_{>1}$, and consider the chain

$$m \to f(m) \to f(f(m)) \to \cdots$$

All powers of m must appear in this chain, so it cannot end in a cycle, hence all terms must be distinct. Now for each $k, \ell \in \mathbb{Z}_{>1}$,

$$f^{f(k)+f(\ell)}(m) = f^{f(\ell)}(m^k) = m^{k\ell} = f^{f(k\ell)}(m),$$

so $f(k) + f(\ell) = f(k\ell)$. However, this implies

$$f(2^{f(3)}) = f(2)f(3) = f(3^{f(2)}),$$

contradicting the fact that f is injective. ■

J2. Find all pairs (a, b) of positive integers such that $(a+1)^{b-1} + (a-1)^{b+1} = 2a^b$.

Proposed by talkon

Answer. $(1, 2)$ and $(4, 2)$

Solution. A simple check gives $(a, b) = (1, 2), (4, 2)$ as the only solutions when $a \leqslant 2$ or $b \leqslant 3$ (when $a = 2$, we use the bound $3^{b-1} + 1 > 2^{b+1}$ for all $b \geqslant 3$ which can be proved by induction).

We will now look for solutions with $a > 2$ and $b > 3$. First, expand and re-write the equation in the form

$$a^{b+1} - (b+3)a^b + \sum_{i=0}^{b-1} a^i \left[\binom{b-1}{i} + (-1)^{b+1-i}\binom{b+1}{i}\right] = 0. \qquad (1)$$

This gives $a \mid \binom{b-1}{0} + (-1)^{b+1}\binom{b+1}{0}$, which is impossible when $b+1$ is even. So, b must be even, and we can divide both sides of (1) by a to get

$$a^b - (b+3)a^{b-1} + \sum_{i=0}^{b-2} a^i \left[\binom{b-1}{i+1} + (-1)^i\binom{b+1}{i+1}\right] = 0. \qquad (2)$$

From (2), we get that

1. $a \mid \binom{b-1}{1} + \binom{b+1}{1} \implies a \mid 2b$; let $2b = at$ where t is a positive integer.
2. $a^2 \mid a\left[\binom{b-1}{2} - \binom{b+1}{2}\right] + 2b \implies a^2 \mid 2b(a-1) - a \implies a \mid t+1 \implies t \geqslant a-1$.

On the other hand, from the initial equation, we get

$$2a^b > (a+1)^{b-1} \implies 2a > \left(1 + \frac{1}{a}\right)^{b-1} \geqslant \left(1 + \frac{1}{a}\right)^{\frac{a(a-1)}{2} - 1} = \left(1 + \frac{1}{a}\right)^{\frac{(a+1)(a-2)}{2}}.$$

But we also have $\left(1 + \dfrac{1}{a}\right)^{a+1} > 1 + (a+1)/a > 2$ and $2^{(a-2)/2} \geqslant 2a$ for all $a \geqslant 11$ (by easy induction). So, we only need to check $3 \leqslant a \leqslant 10$. Modulo 4 eliminates the case when a is odd. From $a \mid t+1$ and b is even, we get that t is odd and so $4 \mid a$. In particular, $a \in \{4, 8\}$.

- If $a = 4$, we have $5^{b-1} + 3^{b+1} = 2 \cdot 4^b$. From $a \mid t+1$, we get that either $t = a - 1 = 3 \implies b = 6$ which is not a solution, or $t \geqslant 2a - 1 = 7 \implies b \geqslant 14$. We'll prove by induction that $5^{b-1} > 2 \cdot 4^b$ for all integer $b \geqslant 14$, which, if true, would gives us contradiction.

  The base case is equivalent to $(5/4)^{13} > 8$, which is true since $(5/4)^{13} = \left(1 + \dfrac{1}{4}\right)^{13} > 1 + 13/4 + \dbinom{13}{2}/4^2 = 73/8 > 8$. The inductive step is easy.

- If $a = 8$, we have $9^{b-1} + 7^{b+1} = 2 \cdot 8^b$. Modulo 3 gives us $1 \equiv 2 \cdot 2^b \pmod 3$. But recall that b is even, which gives $2 \cdot 2^b \equiv 2 \pmod 3$. Contradiction!

In summary, there are no solutions other than those stated at the beginning. ■

J3. There is a calculator with a display and two buttons: $-1/x$ and $x+1$. The display is capable of displaying precisely any arbitrary rational numbers. The buttons, when pressed, will change the value x displayed to the value of the term on the button. (The $-1/x$ button cannot be pressed when $x = 0$.)

At first, the calculator displays 0. You accidentally drop the calculator on the floor, resulting in the two buttons being pressed a total of N times in some order. Prove that you can press the buttons at most $3N$ times to get the display to show 0 again.

Note: partial credit will be given for showing a bound of cN for a constant $c > 3$.

Proposed by talkon

Solution (1: backtracking)**.** Let R and T, for rotate and twist, denote a press of the $x \to -\frac{1}{x}$ and $x \to x+1$ button respectively. For a sequence X of R and Ts, let X^n denote X repeated n times. Clearly, RR is the identity.

LEMMA 1. *For any positive integer n, $RT^nRT(RTT)^{n-1}RT$ is the identity sequence.*

Proof.

$$
\begin{aligned}
x \xrightarrow{RT^n} -\frac{1}{x} + n \xrightarrow{RT} & \frac{1-(n-1)x}{1-nx} \\
\xrightarrow{RTT} & \frac{1-(n-2)x}{1-(n-1)x} \\
\vdots & \\
\xrightarrow{RTT} & \frac{1}{1-x} \\
\xrightarrow{RT} & x. \quad \square
\end{aligned}
$$

LEMMA 2. *For any positive integer n, $T^nRT(RTT)^{n-1}$ sends 0 to 0.*

Proof.

$$
\begin{aligned}
0 \xrightarrow{T^n} n \xrightarrow{RT} & \frac{n-1}{n} \\
\xrightarrow{RTT} & \frac{n-2}{n-1} \\
\vdots & \\
\xrightarrow{RTT} & \frac{1}{2} \\
\xrightarrow{RTT} & 0. \quad \square
\end{aligned}
$$

Now we break off the sequence of N moves into chunks of the form RT^n (except possibly the first chunk which can be in the form T^n). For example,

$$TTTRTRTTTRRTR = T^3(RT^1)(RT^3)(RT^0)(RT^1)(RT^0).$$

Each $RT^0 = R$ can be reversed by $w_0 = R$. By LEMMA 1, each RT^n where $n \geqslant 1$ can be reversed by $w_n = RT(RTT)^{n-1}RT$. By LEMMA 2, the beginning T^n can be reversed by

$u_n = RT(RTT)^{n-1}$ (if $n = 0$ define u_n as the empty sequence). Therefore, a sequence

$$T^b(RT^{a_1})(RT^{a_2})\cdots(RT^{a_n})$$

of $b + n + a_1 + \cdots + a_n = N$ button presses is reversed by the sequence

$$w_{a_n} w_{a_{n-1}} \cdots w_{a_1} u_b$$

of at most

$$(3a_n + 1) + (3a_{n-1} + 1) + \cdots + (3a_1 + 1) + 3b - 1 = 3\left(b + \sum_{i=1}^{n} a_n\right) + n - 1 < 3N$$

button presses. ■

Solution (2: continued fractions)**.** Define functions $A, B, L : \mathbb{Q} \to \mathbb{Z}_{\geqslant 0}$ as follows:

- $A(0) = B(0) = L(0) = 0$.
- For a positive rational number q, note that there is a unique way to write q as a finite simple continued fraction:

$$q = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_m}}}}$$

  where a_0 is a nonnegative integer, and $a_1, a_2, \ldots, a_m$ are positive integers. Define $A(q) = a_0 + a_1 + \cdots + a_m$.

  There is also a unique way to write q as a negative continued fraction:

$$q = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_n}}}}$$

  where $b_0 \geqslant 1$ and $b_1, b_2, \ldots, b_n \geqslant 2$ are integers. Define $B(q) = b_0 + b_1 + \cdots + b_n$, and $L(q) = n$.
- For a negative rational number q, $A(q) = A\left(-\frac{1}{q}\right) + 1, B(q) = B\left(-\frac{1}{q}\right) + 1$, and $L(q) = L\left(-\frac{1}{q}\right)$.

The crucial claim is that $A(q) = B(q) - L(q)$ for all rational q. This can be seen by "induction" through the following claims, both of which are easy to prove:

- For a positive rational number q, $A(q) = B(q) - L(q)$ implies $A(q+1) = B(q+1) - L(q+1)$ and $A\left(-\frac{1}{q}\right) = B\left(-\frac{1}{q}\right) - L\left(-\frac{1}{q}\right)$.
- For a rational number $q > 1$, $A\left(\frac{q-1}{q}\right) = A(q) = B(q) - L(q) = B\left(\frac{q-1}{q}\right) - L\left(\frac{q-1}{q}\right)$.

Next we have the following claims:

- $A(q) = A\left(\frac{1}{q}\right)$; this is obvious.
- If $0 < q < 1$, $A(q) = A(1-q)$. This follows from noting that $1 - q = 1 - \frac{1}{1/q}$ so

$$A(1-q) = B(1-q) - L(1-q) = B\left(\frac{1}{q}\right) - L\left(\frac{1}{q}\right) = A\left(\frac{1}{q}\right) = A(q).$$

Together, these give $A\left(-\frac{1}{x}\right) \leqslant A(x) + 1$ and $A(x+1) \leqslant A(x) + 1$. Therefore if d is the number displayed after the initial N button presses, $A(d) \leqslant N$.

We can use at most one move to make the displayed number d' negative; this gives $A(d') \leqslant N + 1$. Write

$$-d' = d_0 - \cfrac{1}{d_1 - \cfrac{1}{d_2 - \cfrac{1}{\ddots - \cfrac{1}{d_n}}}},$$

so

$$d' = -d_0 + \cfrac{1}{d_1 - \cfrac{1}{d_2 - \cfrac{1}{\ddots - \cfrac{1}{d_n}}}}.$$

It follows that using the move $x \to x + 1$ d_0 times followed by $x \to -1/x$ one time sends us to

$$-d_1 + \cfrac{1}{d_2 - \cfrac{1}{\ddots - \cfrac{1}{d_n}}}.$$

Therefore, repeating this, starting from d' there is a sequence of $d_0 + d_1 + \cdots + d_n + n = B(-d') + L(-d')$ button presses that gives us 0. Finally, as $d_0 \geqslant 1$ and $d_1, \ldots, d_n \geqslant 2$, $B(-d') \geqslant 2L(-d') + 1$. Therefore starting from d, we can press the buttons at most

$$\begin{aligned} 1 + B(-d') + L(-d') &\leqslant 3\left(B(-d') - L(-d')\right) - 1 \\ &= 3A(-d') - 1 = 3\left(A\left(\frac{1}{d'}\right) - 1\right) - 1 = 3A(d') - 4 \leqslant 3N - 1 \end{aligned}$$

times to return to 0. ■

J4. A sequence $a_1, a_2, \ldots$ of positive integers satisfies

$$a_n = \sqrt{(n+1)a_{n-1}+1} \quad \text{for all } n \geqslant 2.$$

What are the possible values of a_1?

Proposed by TacH and ThE-dArK-lOrD

Answer. 1.

Solution. First, the sequence $a_n = n$ obviously satisfies the given condition, so 1 is a possible value of a_1.

Now suppose $a_1 > 1$. We can easily prove by induction on n that $a_n > n$ for all positive integer n. The given relation rearranges to

$$a_n - n = \frac{n+1}{a_n+n} \cdot \big(a_{n-1} - (n-1)\big),$$

so $\{a_n - n\}_{n\geqslant 1}$ is a strictly decreasing sequence of positive integers, but this is impossible. ■

J5. A positive integer $n > 2$ is chosen, and each of the numbers $1, 2, \ldots, n$ is colored red or blue. Show that it is possible to color each subset of $\{1, 2, \ldots, n\}$ either red or blue so that each red number lies in more red subsets than blue subsets, and each blue number lies in more blue subsets than red subsets.

Proposed by ThE-dArK-lOrD

Solution (1: parity)**.** If all numbers $1, 2, \ldots, n$ have the same color, the problem is trivial. Else, WLOG there are $\geqslant 2$ blue and $\geqslant 1$ red numbers.

We start by coloring a subset of $\{1, 2, \ldots, n\}$ blue if it has an odd number of elements, and red otherwise. At this stage, each number is in exactly 2^{n-2} red and 2^{n-2} blue subsets.

Now recolor subsets containing only blue elements blue and subsets containing only red elements red. As there are $\geqslant 2$ blue and $\geqslant 1$ red elements, every number in $\{1, 2, \ldots, n\}$ is in at least one subset that has its color switched. Therefore every number is now in more subsets of its own color than of the other color.

Solution (2: coloring in pairs)**.** We start with a lemma: let $n > 2$, and WLOG n is red. Then, it is possible to color the 2^{n-1} subsets of $\{1, 2, \ldots, n\}$ containing n so that

- For each $i = 1, \ldots, n-1$, of the 2^{n-2} subsets containing both i and n, exactly 2^{n-3} are red and exactly 2^{n-3} are blue.
- Of the 2^{n-1} subsets containing n, exactly $2^{n-2} + 1$ are red.

Proof of lemma. Divide the subsets of $\{1, 2, \ldots, n\}$ containing n into 2^{n-2} pairs of the form $\big(S \cup \{n\}, \{1, \ldots, n\} \setminus S\big)$ where $S \subset \{1, 2, \ldots, n-1\}$. Color 2^{n-3} pairs red (= both sets in the pair are red), 2^{n-3} pairs blue, but make sure that the pair $\big(\{n\}, \{1, \ldots, n\}\big)$ is blue.

For any $i = 1, \ldots, n-1$, each pair has exactly one set with i, so at this point, of the 2^{n-2} subsets containing both i and n, exactly 2^{n-3} is red and exactly 2^{n-3} is blue. Also, each pair has exactly two sets with n, so exactly 2^{n-2} subsets that contain n are currently colored red.

Finally, recolor $\{n\}$ to red, and we're done. $\square$

Proof of J5. Permute the numbers so that 1 and 2 have the same color, say, red. Apply the lemma repeatedly to first color all sets with maximum n, then all sets with maximum $n-1$, and so on until 3, then color all subsets of $\{1, 2\}$ red. ■

Solution (3: induction by two)**.** Suppose there are r red elements $a_1, \ldots, a_r$ and s blue elements $b_1, \ldots, b_s$, with $r + s = n$ and $(r, s) \notin \{(0,0), (1,1)\}$. We will use induction on $\min(r, s)$, with the additional (trivial) constraint that there exists a bichromatic coloring. The cases $0 \in \{r, s\}$ and $(r, s) = (2, 2)$ are easy.

Now consider extending $(r, s) \mapsto (r+1, s+1)$, where $(r, s) \neq (0, 0)$, with new red element a_0 and blue element b_0. Consider a bichromatic coloring $\mathcal{C}$ for S. We construct a coloring for $\{a_0, b_0\} \sqcup S$ as follows: for any $T \subseteq S$, color T with $\mathcal{C}(T)$, $\{a_0\} \sqcup T$ with red, $\{b_0\} \sqcup T$ with blue, and $\{a_0, b_0\} \sqcup T$ with $\mathcal{C}(T)$. It is easy to check that this works. ■

Comments. In Solution 2, it turns out that every number will be in exactly $2^{n-2} + 1$ subsets of its own color.

J6. Determine all positive reals r such that, for any triangle ABC, we can choose points D, E, F trisecting the perimeter of the triangle into three equal-length sections so that the area of $\triangle DEF$ is exactly r times that of $\triangle ABC$.

Proposed by talkon

Answer. $r \in \left[\frac{1}{4}, \frac{4}{9}\right]$.

Solution. First, if $\triangle ABC$ is equilateral, $[DEF]/[ABC]$ is always at least $\frac{1}{4}$ and if $\triangle ABC$ is almost a straight line (sides $1, 1, \varepsilon$), $[DEF]/[ABC]$ is always less than $\frac{4}{9} + \varepsilon'$. Therefore our r must be in $[\frac{1}{4}, \frac{4}{9}]$.

WLOG let $a \geqslant b \geqslant c$, and define $t = \frac{a+b+c}{3}$. Since we can view the area as a function of point D which is continuous, it suffices to show that $[ABC]$ can be both $\leqslant \frac{1}{4}$ and $\geqslant \frac{4}{9}$ of $[DEF]$.

To show that $[DEF]$ can attain a value $\geqslant \frac{4}{9}[ABC]$, choose $D = C$. Clearly $a \geqslant t$. If $b \geqslant t$ then E, F is on CB, CA and

$$\frac{[DEF]}{[ABC]} = \frac{t^2}{ab} > \frac{(a+b)^2}{9ab} \geqslant \frac{4}{9}.$$

Else, E is on CB but F is on AB. In this case $BT = b + c - t$ so

$$\frac{[DEF]}{[ABC]} = \frac{b+c-t}{c} \cdot \frac{t}{a} = \frac{(2b+2c-a)(a+b+c)}{9ac} > \frac{(b+c)(a+b+c)}{9ac} > \frac{(2c)(2a)}{9ac} = \frac{4}{9}.$$

To show that $[DEF]$ can attain a value $\leqslant \frac{1}{4}[ABC]$, choose D on AB, E on AC such that $DE \parallel BC$. Let $AD = x \cdot AB$. Now by the AM-GM Inequality,

$$\frac{[DEF]}{[ABC]} = x(1-x) \leqslant \frac{1}{4}. \qquad \blacksquare$$

Results

We received a total of 24 submissions: 13 for MO3 and 11 for JMO. Here are the scores, with usernames removed for anonymity. A score of – indicates that the user had not submitted a solution for that problem.

InfinityDots **MO3**

Rank	P1	P2	P3	P4	P5	P6	Σ
1	7	7	7	7	6	–	34
2	6	7	7	7	0	3	30
3	6	–	7	7	7	–	27
4	7	7	–	7	3	0	24
5	6	7	–	6	1	0	20
=	6	–	7	7	–	–	20
7	6	3	7	–	–	–	16
8	6	–	1	7	–	–	14
=	7	–	7	0	–	–	14
=	7	0	–	7	0	–	14
11	7	–	–	6	–	–	13
12	6	–	–	–	–	–	6
13	–	0	–	3	–	–	3
Σ	77	31	43	64	17	3	234
Avg.	5.92	2.38	3.31	4.84	1.31	0.23	18.00

InfinityDots **JMO**

Rank	J1	J2	J3	J4	J5	J6	Σ
1	7	7	6	7	7	7	41
2	7	6	7	7	7	–	34
3	7	3	7	1	7	7	32
4	7	1	7	7	7	0	29
5	7	–	7	7	7	0	28
6	7	3	7	7	1	1	26
7	2	–	7	7	7	2	25
8	7	2	7	7	0	0	23
9	2	2	–	2	3	–	9
10	7	–	–	0	–	–	7
11	–	–	6	–	–	–	6
Σ	60	24	61	52	46	17	260
Avg.	5.45	2.18	5.55	4.73	4.18	1.55	23.64

InfinityDots MO3 InfinityDots JMO

Problem Proposers

jeneva
TacH
talkon
ThE-dArK-lOrD
TLP.39

Problem Selection Committee

ArseneLupin
Awien
Continuum
Diamondhead
Gems98
HimeZ
Mr-T19
TacH
talkon
ThE-dArK-lOrD
TLP.39

Graders

Gems98
TacH
talkon
ThE-dArK-lOrD
TLP.39

Testsolvers

anantmudgal09
CantonMathGuy
pieater314159
MarkBcc168
v_Enhance

Participants

AK2654
Allen4567
AnArtist
Anaskudsi
Arhaan
Aryan-23
bangzheng
Cambridge
Chanakya333
CyclicFun
Delta0001
DVDthe1st
e_plus_pi
Electro__Wizard
Elmoliveslong
fastlikearabbit
fermat_theorem
fjm30
Fouad-Almouine
Hamel
hansu
huricane
Hydrogen-Helium
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Liekkas
LITTLE-FERMAT
Math_Magicians
math_pi_rate
MathBoy23
meet18
mfang92
mickeydomath
Mindstormer
MK4J
MNJ2357
p_square
PiMath12345
Pluto1708
primes020
quangminhltv99
Rmo
RMO-prep
roben
rocketscience
sa2001
Sakurafall
SAUDITYA
SHREYAS333
spartacle
Steve12345
Superguy
TheDarkPrince
TRex_1
Tuleuchina
Tumon2001
winnertakeover
Wizard_32
XbenX
yayups

and all private participants