InfinityDots MO3

Language: **English**

Day: **1**

P1. A set of two distinct coprime integers $\{x, y\}$ is said to be a *Pythagorean* if and only if $x^2 + y^2$ is an integer square. Given a Pythagorean, in each move, one can either

(i) change the sign of a number in the Pythagorean, or

(ii) add an integer k to both elements in the Pythagorean so that it is still a Pythagorean.

Show that starting from each Pythagorean, it is possible to reach any Pythagorean in a finite number of moves.

P2. Let $a_1, a_2, a_3, \ldots$ be a nonincreasing sequence of positive real numbers such that

$$a_n \geqslant a_{2n} + a_{2n+1} \text{ for all } n \geqslant 1.$$

Show that there exist infinitely many positive integers m such that

$$2m \cdot a_m > (4m - 3) \cdot a_{2m-1}.$$

P3. In a scalene triangle ABC, the incircle ω has center I and touches side BC at D. A circle Ω passes through B and C and intersects ω at two distinct points. The common tangents to ω and Ω intersect at T, and line AT intersects Ω at two distinct points K and L. Prove that either KI bisects $\angle AKD$ or LI bisects $\angle ALD$.

P4. An $n \times n$ table is written on a square piece of cardboard. Knuffle draws some diagonals in some of the n^2 cells, then uses a knife to cut along the marked diagonals. To Knuffle's surprise, the resulting piece of cardboard is still connected. Show that at least $2n - 1$ cells were left uncut.

P5. Is there a nonempty finite set S of points on the plane that form at least $|S|^2$ harmonic quadrilaterals?
Note: a quadrilateral $ABCD$ is harmonic *if it is cyclic and $AB \cdot CD = BC \cdot DA$.*

P6. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that the set of functions $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$g(a)f(b) + g(b)f(a) \leqslant (a + f(a))(b + f(b)) \text{ for all } a, b \in \mathbb{R}$$

is finite but nonempty.