$\infty^2 \cdot$ MO

Final Report

InfinityDots

April 12, 2018

InfinityDots MO 2 was a Mock IMO contest held on the AoPS Mock Contests forum from March 28 to April 9, 2018.

Contents

Language: **English**

Day: **1**

Problem 1. Determine whether there exists a finite set S of primes such that for all positive integers m, there exists a positive integer n and prime $p \in S$ such that $p^m \mid n!$ but $p^{m+1} \nmid n!$.

Problem 2. Determine all bijections $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f^{f(m+n)}(mn) = f(m)f(n)$$

for all integers m, n.

Note: $f^0(n) = n$, and for any positive integer k, $f^k(n)$ means f applied k times to n, and $f^{-k}(n)$ means f^{-1} applied k times to n.

Problem 3. Let A, B, C be three distinct points on a line ℓ. Prove that for each pair of distinct points B_1, C_1 such that $\overleftrightarrow{B_1C_1}$ does not pass through A, and $\overleftrightarrow{B_1C}$ is not parallel to $\overleftrightarrow{C_1B}$, there is a unique point A_1 satisfying:

(i) A_1 does not lie on $\overleftrightarrow{B_1C_1}$,

(ii) the projections of A onto $\overleftrightarrow{B_1C_1}$, of B onto $\overleftrightarrow{C_1A_1}$, and of C onto $\overleftrightarrow{A_1B_1}$ lie on a line not parallel to ℓ, and

(iii) the reflections of A over $\overleftrightarrow{B_1C_1}$, of B over $\overleftrightarrow{C_1A_1}$, and of C over $\overleftrightarrow{A_1B_1}$ lie on a line not parallel to ℓ.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

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Language: **English**

Day: **2**

Problem 4. Let $P \in \mathbb{Z}[x]$ be a nonconstant polynomial without integral roots. Prove that there is a positive integer $m \leqslant 3 \cdot \deg P$ such that $P(m)$ does not divide $P(m+1)$.

Problem 5. Let $c_1, c_2, \ldots, c_k$ be integers. Consider sequences $\{a_n\}$ of integers satisfying

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

for all $n \geqslant k+1$. Prove that there is a choice of initial terms $a_1, a_2, \ldots, a_k$ not all zero satisfying: there is an integer b such that p divides $a_p - b$ for all primes p.

Problem 6. Ana has an $n \times n$ lattice grid of points, and Banana has some positive integers $a_1, a_2, \ldots, a_k$ which sum to exactly n^2. Banana challenges Ana to partition the n^2 points in the lattice grid into sets $S_1, S_2, \ldots, S_k$ so that for all $i \in \{1, 2, \ldots, k\}$,

(i) $|S_i| = a_i$, and

(ii) the set S_i has an axis of symmetry.

Prove that Ana can always fulfill Banana's challenge.

Note: a line ℓ is said to be an axis of symmetry of a set S if the reflection of S over ℓ is precisely S itself.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Solutions and Comments

Overall Comments.

On difficulty, we feel that our problems this year are generally easier than last year (especially P5), but still on a Mock IMO level.

On subject balancedness, we decided to go with a single hard geometry problem as we feel geometry is declining as a subject, and that problems in other subjects are more interesting. Also, in our problem selection process, we decided to rate each problem by its "subject-ness" which sum to 1. We'd also like to report that we rated our contest as follows:

Algebra: 1.88, Combinatorics: 1.47, Geometry: 1.00, Number Theory: 1.65

which we find balanced enough.

Problem 1.

Determine whether there exists a finite set S of primes such that for all positive integers m, there exists a positive integer n and prime $p \in S$ such that $p^m \mid n!$ but $p^{m+1} \nmid n!$.

Proposed by TacH

Answer. No.

Solution. We will prove, by induction on n, the following stronger statement: for any set of n primes $\{p_1, p_2, \cdots, p_n\}$, there exists an integer A such that for any A consecutive integers, there is at least one not in $\bigcup S_i$, where S_i is defined as $\{\nu_{p_i}(m!) \mid m \in \mathbb{Z}^+\}$.

The base case is clear: choose $A = p + 1$. For the inductive step, it suffices to exhibit a new value A' for $\{p_1, p_2, \ldots, p_{n+1}\}$ in terms of the value of A for $\{p_1, p_2, \cdots, p_n\}$.

We claim $A' = Ap_{n+1}^{A+1} + A$ suffices. Consider A' consecutive integers. If the last $A' - A$ does not contain A consecutive integers not in S_{n+1}, then S_{n+1} must have at least p_{n+1}^{A+1} numbers in that range (one for each A consecutive integers), representing p_{n+1}^{A+1} consecutive values of m. One such m must be divisible by p_{n+1}^{A+1}, and $[\nu_{p_{n+1}}(m!) - A, \nu_{p_{n+1}}(m!) - 1]$ will be our desired interval.

Therefore, any A' consecutive integers must contain A consecutive integers not in S_{n+1}, so using the property of A, at least one of them must be not in $\bigcup_{i=1}^{n+1} S_i$ as required. ■

Comments. A better A' in the inductive step is

$$A + \nu_{p_{n+1}}(p_{n+1}^{A+1}!) = A + p_{n+1}^{A} + p_{n+1}^{A-1} + \cdots + p_{n+1} + 1,$$

but this is a bit harder to prove.

Problem 2.

Determine all bijections $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f^{f(m+n)}(mn) = f(m)f(n)$$

for all integers m, n.

Note: $f^0(n) = n$, *and for any positive integer* k, $f^k(n)$ *means* f *applied* k *times to* n, *and* $f^{-k}(n)$ *means* f^{-1} *applied* k *times to* n.

Proposed by talkon

Answer. The infinite family of functions $n \mapsto n + c$ for any integer c, and the function $n \mapsto (-1)^{n+1}n$.

Solution. We consider two cases, depending on the value of $f(0)$.

Case 1: $f(0) = 0$.

By plugging in $(m, n) = (k, -k)$, we have $-k^2 = f(k)f(-k)$ for all integers k. Since f is bijective, by induction on k, $\{f(k), f(-k)\} = \{k, -k\}$ for all positive integers k. Hence $f(f(n)) = n$ for all integers n.

Now suppose that m, n are integers with the same parity, so $f(m + n)$ is even. Hence,

$$mn = f^{f(m+n)}(mn) = f(m)f(n),$$

so either both $f(m) = m$ and $f(n) = n$ or $f(m) = -m$ and $f(n) = -n$. Therefore there are four solutions left to check: $n \mapsto n, n \mapsto -n, n \mapsto (-1)^n n$, and $n \mapsto (-1)^{n+1}n$, and by considering m even and n odd, we can see that only two work: $n \mapsto n$ and $n \mapsto (-1)^{n+1}n$.

Case 2: $f(0) \neq 0$.

Plug in $(m, n) = (f^{-1}(k), 0)$ to get $f^k(0) = kf(0)$ for all integers k. In particular, when $k = -1$ we have $f^{-1}(0) = -f(0)$. Now substitute in $(m, n) = (m, -f(0))$ to get, for all integers m,

$$f^{f(m-f(0))}(-mf(0)) = 0. \tag{1}$$

Now note that the orbit $\ldots \to f^{-2}(0) \to f^{-1}(0) \to 0 \to f(0) \to f(f(0)) \to \ldots$ contains all multiples of $f(0)$, so it is unbounded and not periodic. Hence from

$$\begin{aligned} f^{f(n)-n-f(0)}(0) &= f^{f(n)}\big(f^{-n-f(0)}(0)\big) \\ &= f^{f((n+f(0))-f(0))}\big((-n-f(0)) \cdot f(0)\big) \\ &= 0 \end{aligned}$$

where the second equation follows from $f^k(0) = kf(0)$ and the third equation follows from (1), we have $f(n) - n - f(0) = 0$ for all integers n. Hence the function f must be of the form $n \mapsto n + c$ for some constant c, and it's easy to see that all such functions work. ■

Comments. There are several possible ways to proceed in *Case 2*. For example, another way is to plug in $m = 0, f(0)$ and $2f(0)$.

Problem 3.

Let A, B, C be three distinct points on a line ℓ. Prove that for each pair of distinct points B_1, C_1 such that $\overleftrightarrow{B_1C_1}$ does not pass through A, and $\overleftrightarrow{B_1C}$ is not parallel to $\overleftrightarrow{C_1B}$, there is a unique point A_1 satisfying:

(i) A_1 does not lie on $\overleftrightarrow{B_1C_1}$,

(ii) the projections of A onto $\overleftrightarrow{B_1C_1}$, of B onto $\overleftrightarrow{C_1A_1}$, and of C onto $\overleftrightarrow{A_1B_1}$ lie on a line not parallel to ℓ, and

(iii) the reflections of A over $\overleftrightarrow{B_1C_1}$, of B over $\overleftrightarrow{C_1A_1}$, and of C over $\overleftrightarrow{A_1B_1}$ lie on a line not parallel to ℓ.

Proposed by TacH

Solution. Let $A_\perp, B_\perp, C_\perp$ be the projections of A onto $\overleftrightarrow{B_1C_1}$, of B onto $\overleftrightarrow{C_1A_1}$, and of C onto $\overleftrightarrow{A_1B_1}$, and let A', B', C' be the reflections of A over $\overleftrightarrow{B_1C_1}$, of B over $\overleftrightarrow{C_1A_1}$, and of C over $\overleftrightarrow{A_1B_1}$ respectively.

Since the lines in (ii) and (iii) are not parallel to ℓ, there exist a spiral similarity λ sending $A \to A'$, $B \to B'$, and $C \to C'$. Let S be the center of λ, let $\ell' = \overleftrightarrow{B_1C_1}$, and let $T = \ell \cap \ell'$.

Note that there is also a spiral similarity χ with center S that sends $A \to B$ and $A_\perp \to B_\perp$. Let the tangent to (ABS) at B cut $C_1B_\perp$ at P. Since $\angle A_\perp AB = \angle B_\perp BP$, $\triangle AA_\perp T \sim \triangle BB_\perp P$. Therefore χ sends $T \to P$. Hence, from $B = AT \cap BP$, $(BTPS)$ is cyclic.

Now, S is also the center of a spiral similarity ψ that sends $T \to A_\perp$ and $P \to B_\perp$. Since $C_1 = TA_\perp \cap PB_\perp$, $(STPC_1)$ is cyclic.

Combining this with the above, we have now shown that S must be on (BTC_1). Similarly, S must be on (CTB_1) as well, hence S is unique, and is the intersection of (BTC_1) and (CTB_1).

Finally, from this unique S, we can construct A_1 as the intersection of $B_\perp C_1$ and $C_\perp B_1$. Clearly this A_1 works, and it must be unique as well. ■

Comments.

1. The conditions in the problem are needed; if A is on $\overleftrightarrow{B_1C_1}$ then ℓ_1, ℓ_2 found in the above solution will coincide with ℓ, and if $\overleftrightarrow{B_1C}$ is parallel to $\overleftrightarrow{C_1B}$, the locus becomes a line since $B_\perp, C_1, C_\perp, B_1$ lie on the same line.

2. If we remove the *not parallel to* ℓ condition, there may be four additional points A formed, which can be constructed as follows. First, draw a line ℓ' through $A_\perp$ parallel to ℓ. Then, draw the circle Γ_B with diameter BC_1 possibly cutting ℓ at X_1, X_2, and the circle Γ_C with diameter CB_1 possibly cutting ℓ at Y_1, Y_2. Then the additional points are the points formed by the intersection of C_1X_i and B_1Y_j.

Problem 4.
Let $P \in \mathbb{Z}[x]$ be a nonconstant polynomial without integral roots. Prove that there is a positive integer $m \leqslant 3 \cdot \deg P$ such that $P(m)$ does not divide $P(m+1)$.

Proposed by ThE-dArK-lOrD and tastymath75025

Solution. Let u be the least positive integer such that $P(u) \neq \pm P(1)$. Since $P(x) = \pm P(1)$ can only have $2n$ roots, $2 \leqslant u \leqslant 2n+1$.

Using finite differences or Lagrange Interpolation, we have

$$P(u+n) = \sum_{i=0}^{n} (-1)^{n-i+1} \binom{n}{i} P(u-1+i).$$

If $P(u) \mid P(u+1) \mid \cdots \mid P(u+n)$, the above equation reduces to

$$P(u+n) \equiv \pm P(u-1) \equiv \pm P(1) \pmod{P(u)}$$

hence $P(u) \nmid P(u+n)$ which is a contradiction. Therefore, $P(m) \nmid P(m+1)$ for some $m \in \{u, u+1, \ldots, u+n-1\}$. ■

Comments.

1. For the record, we'd like to note that a more general (and unsolved) version of this problem was actually posted in the High School Olympiad forums by @ThE-dArK-lOrD:

   Given a positive integer n. Determine the largest positive integer m such that there exists a polynomial $P \in \mathbb{Z}[x]$ with degree n such that $P(i) \mid P(i+1)$ for all positive integer $i < m$.

   The post has been deleted, and since there were no comments, exposure was very low. @tastymath75025 has provided a bound $n+2 \leqslant m \leqslant 3n$ privately, from which we have created this problem. Note that @tastymath75025 is not a member of InfinityDots.

2. The lower bound $n+2 \leqslant m$ in the above comment can be shown by taking $P(x) = 2x-1$ when $n=1$, and taking

   $$P(x) = (x-1)(x-2)\cdots(x-n) + \frac{n!}{n-1}$$

   otherwise.

   When $n = 2k$ is even, we can also show that $n+3 \leqslant m$ by considering a polynomial $Q(x)$ of degree k satisfying $Q((t+1/2)^2) = \pm 1$ for all $t = 0, 1, \ldots, k$, then take $P(x) = cQ((x-k-3/2)^2)$ where c is a big enough constant to make $P \in \mathbb{Z}[x]$.

3. One way to get a better lower/higher bound on the general problem is to find the exact maximum value of m' where there exists a $P \in \mathbb{Z}[x]$ such that $|P(1)| = |P(2)| = \cdots = |P(m')|$. It will immediately follow that $m'+1 \leqslant m \leqslant m'+n$.

Problem 5.

Let $c_1, c_2, \ldots, c_k$ be integers. Consider sequences $\{a_n\}$ of integers satisfying

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

for all $n \geqslant k+1$. Prove that there is a choice of initial terms $a_1, a_2, \ldots, a_k$ not all zero satisfying: there is an integer b such that p divides $a_p - b$ for all primes p.

Proposed by talkon

Solution (1: Algebraic)**.** If $c_1, c_2, \ldots, c_k$ are all zero, choose $a_i = k!$ for all $i = 1, 2, \ldots, k$. The recurrence relation gives us $a_n = 0$ for all $n \geqslant k+1$. Not hard to see that this gives $p \mid a_p \implies a_p \equiv 0 \pmod p$ for all primes p.

Now, suppose $c_1, c_2, \ldots, c_k$ are not all zero. Let $z_1, z_2, \ldots, z_k$ be the roots (counting multiplicities) of the characteristic equation $\lambda^k - \sum_{i=1}^{k} c_i \lambda^{k-i} = 0$. The key part is choosing

$$a_n = z_1^n + z_2^n + \cdots + z_k^n$$

for all n.

By Vieta, we get that the elementary symmetric polynomials

$$e_j = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le k} z_{i_1} z_{i_2} \ldots z_{i_j} = (-1)^{j+1} c_j$$

is an integer for all $j = 1, 2, \ldots, k$. By the Fundamental Theorem of Symmetric Polynomials, we get that $P(z_1, z_2, \ldots, z_k)$ is an integer for all symmetric polynomials $P \in \mathbb{Z}[x_1, x_2, \ldots, x_k]$. In particular, a_n is an integer for all $n \in \mathbb{Z}^+$.

To prove that $a_1, a_2, ..., a_k$ are not all zero, note that if $a_i = 0$ for all $i = 1, 2, \ldots, k$, we can use Newton's identities to prove by induction on i that $\sigma_i = 0$ for all $i = 1, 2, \ldots, k$. This implies $c_i = 0$ for all $i = 1, 2, \ldots, k$, which is a contradiction.

Now, for each prime p, we've

$$c_1^p = (z_1 + z_2 + ... + z_k)^p = \sum_{i=1}^{k} z_i^p + pT(z_1, z_2, ..., z_k) = a_p + pT(z_1, z_2, ..., z_k)$$

for some symmetric polynomial $T \in \mathbb{Z}[x_1, x_2, ..., x_k]$. So, $T(z_1, z_2, ..., z_k)$ is an integer. This gives $a_p \equiv c_1^p \equiv c_1 \pmod p$, so we are done by choosing $b = c_1$. ■

Solution (2: Combinatorial)**.** If c_i are all zero, then just choose a_i like in the first solution. Else, consider necklaces of size n that is

- marked, that is, we mark the positions on it as $1, 2, \ldots, n$, and
- created from parts of size $1, 2, \ldots, k$.

We assign a value to each necklace defined as the product of c_i's where i's are the size of the parts counting multiplicity.

Now define a_n as the sum of values of all such necklaces of size n. Clearly a_n is an integer for all n, and if $j = \min\{i \mid c_i \neq 0\}$ then $a_j = jc_j$ is clearly nonzero. Next we'll show that $\{a_n\}$ satisfy the recurrence relation.

Let's look at each such necklace. The position 1 of the necklace is the r^{th} position (counting clockwise) of a part of size i, where $1 \leqslant r \leqslant i \leqslant k$. Call (r, i) the *type* of that necklace.

By looking at the size of part containing position $n + 1 - r$ (that is the part immediately counterclockwise from the part containing position 1), we can see that the sum of values of necklaces of size n with type (r, i), denoted by $a_{n,(r,i)}$, satisfies

$$a_{n,(r,i)} = c_1 a_{n-1,(r,i)} + c_2 a_{n-2,(r,i)} + \cdots + c_k a_{n-k,(r,i)}$$

when $n > k$. Summing the above over all types (r, i) implies $\{a_n\}$ satisfies the recurrence relation.

Finally, we can see that actually, when $n = p$ is prime, we can remove the marks from the necklace allowing it to be rotated into p positions. Except when all parts have size 1, where rotation doesn't change the necklace, these p positions are all counted as distinct in a_p, but have the same value. Since the necklace where all parts have size 1 has value c_1^p, it follows that for all prime p,

$$s_p \equiv c_1^p \equiv c_1 \pmod{p}$$

hence we can choose $b = c_1$. ■

Comments. The sequences $\{a_n\}$ in the two solutions actually turn out to be exactly the same sequence. One way to show this is by noting that the equation about $a_{n,(r,i)}$ is still true for all $n \leqslant k$, (taking $a_{n,(r,i)} = 0$ for all $n < i$) except when $n = i$ where the LHS is c_i and the RHS is 0. This actually implies that

$$a_n = nc_n + \sum_{j=1}^{n-1} c_j a_{n-j}$$

for all $n \in \mathbb{Z}^+$ (taking $c_n = 0$ for all $n \geqslant k$), which is exactly the same as Newton's identities

$$p_n = (-1)^{n+1} n e_n + \sum_{j=1}^{n-1} (-1)^{j+1} e_j p_{n-j}$$

where p_n is the *power sum symmetric polynomial* $\sum_{i=1}^{k} z_i^n$. Therefore, by induction, $a_n = p_n$ for all n, so the two solutions result in the same sequence a_n.

Problem 6.
Ana has an $n \times n$ lattice grid of points, and Banana has some positive integers $a_1, a_2, \ldots, a_k$ which sum to exactly n^2. Banana challenges Ana to partition the n^2 points in the lattice grid into sets $S_1, S_2, \ldots, S_k$ so that for all $i \in \{1, 2, \ldots, k\}$,

(i) $|S_i| = a_i$, and

(ii) the set S_i has an axis of symmetry.

Prove that Ana can always fulfill Banana's challenge.

Note: a line ℓ is said to be an axis of symmetry of a set S if the reflection of S over ℓ is precisely S itself.

Proposed by talkon

Solution. Let's start with some terminology and notations. For collections C, D of positive integers, and sets S of lattice points,

- $C + D$ means the collection created by adding C and D. That is, an element e that appears c times in C and d times in D will appear $c + d$ times in $C + D$.
- $C \subset D$ means every element $e \in C$ appears at least as much in D as in C.
- if $C \subset D$, then $D - C$ is the collection created by subtracting C from D. That is, an element e that appears c times in C and d times in D will appear $d - c$ times in $D - C$.
- $s(C)$ is the sum of elements of C counting multiplicity.
- E_C, U_C, P_C are subcollections with $E_C + U_C + P_C = C$ where E_C ('even') contains only even numbers, U_C ('unpaired odds') contains each odd number at most once, and P_C ('paired odds') contains each odd number an even times. Note that E_C, U_C, P_C are uniquely defined for each C.
- $P_C/2$ is the collection that is exactly half of P_C. In formal terms, $P_C/2$ is the collection that $P_C/2 + P_C/2 = P_C$
- We call (S, C) *colorable* iff $|S| = s(C)$ and it's possible to partition S into k sets $S_1, \ldots, S_k$ so that (i) for each j, $|S_j| = c_j$, and (ii) the set S_j has an axis of symmetry ℓ_j.

We'll need the following easy lemma:

Lemma 1. If S is a set of lattice points with an axis of symmetry ℓ_S so that t points of S lies on ℓ_S, and C is a collection of positive integers summing to $|S|$ with at most t odd numbers, then (S, C) is colorable. Furthermore, there exists a (proper) coloring where all the axes ℓ_j coincide with ℓ_S.

Proof. Divide S into t isolated points on ℓ_S and $(|S| - t)/2$ pairs of points symmetric w.r.t. ℓ_S. For each even c_j, choose some $c_j/2$ pairs, while for each odd c_j, choose an isolated point and some $(c_j - 1)/2$ pairs. □

We will now prove the following statement by induction on n, which clearly implies the problem:

<u>Statement:</u> Let T_n denote the $n \times n$ lattice grid $\{1, 2, \dots, n\}^2$ and $A = \{a_1, a_2, \dots, a_k\}$ be a collection of positive integers summing to n^2. Then, (T_n, A) is colorable even with the following added condition:

(iii) for any $a_k \in (E_A + U_A)$, the axis ℓ_k is the line $x = y$.

Proof. The base case $n = 1$ is obvious. Suppose that the above statement is true for all $n \leqslant m - 1$. We'll show that it's true for $n = m$ as well.

I. <u>Adjustment of A</u>

If A contains only even numbers, the problem is trivial by Lemma 1. Else, we can add all elements of E_A to the maximal odd $a_{\max}$ to create new collection A' that contains only odd elements. If $E_A \neq \emptyset$, $a_{\max} + s(E_A)$ is always unique in A' (and hence is in U'_A), hence by Lemma 1, any proper coloring of A' can be transformed to that of A as well, and the even numbers will have $x = y$ as the axis as required. So, from here onward, we'll consider only the case when A contains only odd numbers, i.e. $E_A = \emptyset$. From now on, for brevity, we will omit the subscript A in U_A, P_A.

For any two possible sets U, U' that $|U| = |U'|$ and $s(U) = s(U')$, since the axis of all S_is that $a_i \in U$ is always the same and by Lemma 1, it's possible to recolor all points when U is changed to U'. Hence we can assume that U is of the form

$$\{1, 3, 5, \dots, 2u - 3, 2u - 1\} \cup W$$

where u is the maximum possible that makes W an empty set or $\{2u + 2\gamma - 1\}$ for some $\gamma > 1$. Note that any set V that $s(V) \geqslant |V|^2$ can be transform to that form.

Now, let $p_{\max}$ be the largest element in $P/2$. If $p_{\max} \geqslant 2u + 3$, we can change the pair $p_{\max}, p_{\max}$ from P to $2u+1, 2p_{\max} - (2u+1)$ making them both new distinct unique members, and so belong to set U' of new collection. We've the following cases

- If W is an empty set, $s(U') = u^2 + 2p_{\max} \geq u^2 + 4u + 6, |U'| = u + 2$.
- If $W = \{2u + 2\gamma - 1\}$, $s(U') = u^2 + (2u + 2\gamma - 1) + 2p_{\max} \geq u^2 + 6u + 9, |U'| = u + 3$.

In both cases, we get $s(U') \geq |U'|^2$, hence we can transform it to the above form. In this new collection, the new maximal odd in $P/2$ is at most $p_{\max}$, and the new $2u' - 1$ of U' is more than the old $2u - 1$, so this reduces the value $p_{\max} - (2u - 1)$. Hence, we can make an additional assumption that $p_{\max} \leqslant 2u + 1$.

So, now our situation reduces to when A is in this form:

	1	3	$\cdots$	$2u-1$	$2u+1$	$2u+3$	$2u+5$	$\cdots$
U	/	/	$\cdots$	/	$\emptyset$	$\emptyset$ or only $2u+2\gamma-1$		
$P/2$			maybe				$\emptyset$	

II. We list some possible ways to reduce to a smaller case for induction

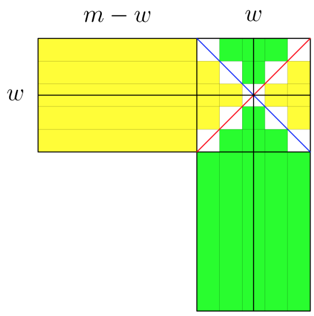
Lemma 2. If there exists an odd integer w, and collections R, X of odd integers such that

(a) $R \subset P/2$ and $2R + X \subset A$,

(b) $|R| \leqslant m - 1$ and $|X| = w$, and

(c) $2s(R) + s(X) \geqslant w(2m - w) \geqslant 2s(R) + 2w - 1$,

then the Statement is true for (T_n, A).



Proof. Consider the above figure (note that squares in the figure represent points), and use this procedure:

Step 1. Use the lemma to put a copy of R on the yellow points, using the horizontal black line as the axis.

Step 2. Use the lemma to put the other copy of R on the mirroring green points, using the vertical black line as the axis.

Step 3. To each odd in X, assign a point each on the red diagonal.

Step 4. Assign leftover pairs of points (with each pair symmetric w.r.t the line $x = y$) to a random odd in X. Repeat this until all points in the figure (points with $x > m - w$ or $y > m - w$) are assigned.

After the procedure, all odds in R are used up, and all odds in X will have been reduced to evens. Hence any unused member of U is still in the new U for $n = m - w$. Now apply the inductive hypothesis and we're done. □

Corollary 3. ($w = 1$ case of Lemma 2.) If there exists $o \in A$ and a collection R that

(a) $R \subset P/2$ and $2R + \{o\} \subset A$,

(b) $s(R) \leqslant m - 1$, and

(c) $2s(R) + o \geqslant 2m - 1$,

then the Statement is true for (T_n, A).

Lemma 4. If there is a collection $\{p, p, o_1, o_2\}$ of odd numbers which is a subcollection of A and satisfies $2p + o_1 + o_2 \geqslant 4m - 4$ then the Statement is true for (T_n, A).

Proof. Put the two copies of p on the border on the $(1, m)$ and $(m, 1)$ corners then assign $(1, 1), (m, m)$ to o_1 and o_2 respectively. Use pairs left from o_1, o_2 to fill the whole border. Any leftover are even and covered by the inductive hypothesis. □

III. We consider a certain maximal subcollection of $P/2$

Consider a subcollection $Q \subset P/2$ that maximises $s(Q) < m$. Let $t = m - s(Q)$. We make the following observations:

- If there's an element $a_{\text{big}} \geqslant 2t - 1$ in $A - 2Q$, we can use Corollary 3 with $(o, R) = (a_{\text{big}}, Q)$. Else, all elements in U are at most $2t - 3$, which gives

$$2t - 3 \geqslant 2u(+2\gamma) - 1 \implies t \geqslant u(+1) + 1 = |U| + 1.$$

- If there's an element $a_{\text{small}} < t$ in $P/2 - Q$ then $Q + \{a_{\text{small}}\}$ should be Q instead, a contradiction. Hence, all elements in $P/2 - Q$ are at least t.

IV. We finish the problem.

First, we divide the possible value of t into two cases.

Case 1: $t > m/2$. Since $t > m - t$, if $P/2$ contains any element a that $t < a < m$, the set $\{a\}$ must be Q instead, a contradiction. Hence all elements in $P/2 - Q$ are at least m. We, then, have the following cases:

- If $P/2$ is empty set then it follows that there are at most m odd numbers in $A = U$ then Lemma 1 finish the problem.
- If $P/2$ contains at least two numbers at least m then we can use Lemma 4.
- Else $P/2 - Q$ contain at most one element which must be less than $2t - 1$, this gives

$$s(P/2) \leqslant (m - t) + (2t - 3) = m + t - 3 \leqslant 2m - 4 \implies s(P) \leqslant 4m - 8,$$

so $s(U) \geqslant m^2 - 4m + 8$. If U contains no element that is at least $2m - 1$, we get $s(U) \leqslant 1 + 3 + 5 + ... + (2m - 3) = (m - 1)^2$. But $s(U) \equiv m^2 \pmod 2$, this means $|U|$ can't be $m - 1$, and so $s(U) \leqslant (m - 1)^2 - (2m - 5) = m^2 - 4m + 4$ (in case that $2m - 3$ is equal to $2u + 2\gamma - 1$), contradiction with $s(U) \geqslant m^2 - 4m + 8$. Hence U contains an element a that is at least $2m - 1$. We, then, use Corollary 3 with $(o, R) = (a, \emptyset)$.

Case 2: $t \leqslant m/2$. We divide into further subcases based on $|U|$ and $p_{\max}$. Recall from **I.** that we have $p_{\max} \leqslant 2u + 1 \leqslant 2|U| + 1$.

When $p_{\max} \leqslant 5$ or when $(|U|, p_{\max}, m) = (3, 7, 8)$, it is not hard to check that in these cases we either have enough 5's to use Lemma 2 with $w = 5$ and R, X containing only 5's, or can use Corollary 3.

In all other cases, we can use Lemma 2 as follows:

Case	w	X	R
$\lvert U\rvert$ odd, $p_{\max} \leqslant (\lvert U\rvert-1)^2+1$	$\lvert U\rvert$	U	
$\lvert U\rvert$ even, $p_{\max} \leqslant (\lvert U\rvert-2)^2+1$	$\lvert U\rvert-1$	$U-\{1\}$	
$(\lvert U\rvert, p_{\max}) = (3,7), m \geqslant 9$	5	$U+\{7,7\}$	
$(\lvert U\rvert, p_{\max}) = (4,7)$	5	$U+\{7,7\}-\{1\}$	see below
$(\lvert U\rvert, p_{\max}) = (4,9)$	5	$U+\{9,9\}-\{1\}$	
$(\lvert U\rvert, p_{\max}) = (5,11)$	7	$U+\{11,11\}$	
$(\lvert U\rvert, p_{\max}) = (6,11)$	7	$U+\{11,11\}-\{1\}$	
$(\lvert U\rvert, p_{\max}) = (6,13)$	7	$U+\{13,13\}-\{1\}$	

To create R, we randomly choose an element from $F = P/2 - Q - (P/2 \cap X)$ (named F for 'finally the end of the solution is near') to put in R until $s(R) \geqslant \frac{w(2m-w)-s(X)}{2}$ then show that it's in the range in condition (c) of Lemma 2. The constraints we need to verify is that

$$\max F \leqslant \frac{s(x)-(2w-1)}{2}+1$$

and

$$s(F)+s(X) \geqslant w(2m-w).$$

One can check that for the cases in the table above, these equations hold, using the facts that $\max F \leqslant p_{\max}$, $m \geqslant 2t \geqslant 2|U|+2$, and $s(F)+s(X) \geqslant m^2+(-1)-2s(Q)$, with (-1) only in cases that we removed $\{1\}$ from U. ■

Comments. 1. The following, due to @DVDthe1st, is a simple solution for odd n:

Roughly speaking, we can WLOG all a_i odd (since we can always pair even terms with any odd one, then extract it out later via symmetry). Since $1+3+...+(2n-1) = n^2$, there are at most n distinct values. Out of all the distinct values, pick out one a_i from those with an odd number of such values (call this set S) and pair the remaining. Now, note the n cells along the middle row. For each a_i in set S, we split it by putting one cell on the middle row, and then we aim to place $\frac{a_i-1}{2}$ in both the top half and the bottom half. If $|S| < n$, we add in pairs of equal a_i until the middle row is used up entirely. The remaining pairs we allocate one each to the top half and the bottom half. So it becomes the problem of splitting the top half and bottom half into a new set of a_i, and we can somewhat induct this.

Upon further thoughts, my solution for P6 can be extended to all odd n. Essentially, we need to assume that the extra a_i that were used on the middle row are maximal, so if we use 1's then the remaining pairs are all 1's and the inducive step is trivial. Otherwise, we realise that the split a_i parts don't have to be symmetric within the top half, so they become essentially "spares", which we can use to eliminate one row if we so wish.

2. The above idea can be generalized to prove that the result is true when $n \times n$ is changed to $m \times n$ for $m \geqslant n$ and n odd. It's also possible that a full solution to the problem can be found from this idea.

Results

In the end, a total of 12 users submitted solutions. Here are the scores, with usernames removed for anonymity. A score of – indicates that the user had not submitted a solution for that problem.

Rank	P1	P2	P3	P4	P5	P6	Σ
1	7	7	7	7	7	–	35
2	7	7	2	7	7	1	31
3	7	7	–	7	6	3	30
4	0	7	7	7	6	1	28
5	–	7	7	7	5	–	26
6	–	7	–	7	6	–	20
7	–	7	–	3	7	–	17
8	5	7	–	–	–	–	12
9	–	–	–	7	–	–	7
9	–	7	–	–	–	–	7
11	0	–	–	–	6	–	6
12	0	–	–	–	–	–	0
Σ	26	63	23	52	50	5	219
Avg.	2.17	5.25	1.92	4.33	4.17	0.42	18.25

Final Remarks

InfinityDots MO 2 has concluded. Thanks to everyone who has been a part of the contest.
Problem proposers: TacH, talkon, tastymath75025, ThE-dArK-lOrD.
PSC Members: ArseneLupin, Awien, Continuum, Diamondhead, Gems98, HimeZ, Mr-T19, TacH, talkon, ThE-dArK-lOrD.
Graders and Outreach: TacH, talkon, ThE-dArK-lOrD.
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This report is compiled by talkon.