∞²·MO

Language: **English**

Day: **1**

Problem 1. Determine whether there exists a finite set S of primes such that for all positive integers m, there exists a positive integer n and prime $p \in S$ such that $p^m \mid n!$ but $p^{m+1} \nmid n!$.

Problem 2. Determine all bijections $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f^{f(m+n)}(mn) = f(m)f(n)$$

for all integers m, n.

Note: $f^0(n) = n$, and for any positive integer k, $f^k(n)$ means f applied k times to n, and $f^{-k}(n)$ means f^{-1} applied k times to n.

Problem 3. Let A, B, C be three distinct points on a line ℓ. Prove that for each pair of distinct points B_1, C_1 such that $\overleftrightarrow{B_1C_1}$ does not pass through A, and $\overleftrightarrow{B_1C}$ is not parallel to $\overleftrightarrow{C_1B}$, there is a unique point A_1 satisfying:

(i) A_1 does not lie on $\overleftrightarrow{B_1C_1}$,

(ii) the projections of A onto $\overleftrightarrow{B_1C_1}$, of B onto $\overleftrightarrow{C_1A_1}$, and of C onto $\overleftrightarrow{A_1B_1}$ lie on a line not parallel to ℓ, and

(iii) the reflections of A over $\overleftrightarrow{B_1C_1}$, of B over $\overleftrightarrow{C_1A_1}$, and of C over $\overleftrightarrow{A_1B_1}$ lie on a line not parallel to ℓ.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

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Day: **2**

Problem 4. Let $P \in \mathbb{Z}[x]$ be a nonconstant polynomial without integral roots. Prove that there is a positive integer $m \leqslant 3 \cdot \deg P$ such that $P(m)$ does not divide $P(m+1)$.

Problem 5. Let $c_1, c_2, \ldots, c_k$ be integers. Consider sequences $\{a_n\}$ of integers satisfying

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

for all $n \geqslant k+1$. Prove that there is a choice of initial terms $a_1, a_2, \ldots, a_k$ not all zero satisfying: there is an integer b such that p divides $a_p - b$ for all primes p.

Problem 6. Ana has an $n \times n$ lattice grid of points, and Banana has some positive integers $a_1, a_2, \ldots, a_k$ which sum to exactly n^2. Banana challenges Ana to partition the n^2 points in the lattice grid into sets $S_1, S_2, \ldots, S_k$ so that for all $i \in \{1, 2, \ldots, k\}$,

(i) $|S_i| = a_i$, and

(ii) the set S_i has an axis of symmetry.

Prove that Ana can always fulfill Banana's challenge.

Note: a line ℓ is said to be an axis of symmetry of a set S if the reflection of S over ℓ is precisely S itself.

Time: 4 hours and 30 minutes
Each problem is worth 7 points