$\boxed{\infty \cdot \text{MO}}$ Official Solutions

InfinityDots

April 4, 2017

Problem 1. We have a deck of $mn \geq 3$ cards. A *weird shuffle* is a process of shuffling a deck of cards following these steps:

1. Pull out the first m cards of the deck, making a pile. Repeat this until we get n piles.
2. Arrange the new deck by placing the top card of pile 1 to the bottom of the new deck, following by the top card of pile 2, and that of pile 3. Continue doing this until placing the top card of the pile n, and then go back to that of pile 1. Repeat these processes until the cards in those n piles run out.

Show that the deck will return to the initial position in at most $mn - 1$ weird shuffles.

Solution. Call the positions of cards from top to bottom $0, 1, 2, \ldots, mn-1$. The top- and bottom-most cards will always switch places, so they will return to the initial position in 2 shuffles.

Now suppose that a card start at position $k \not\equiv 0 \pmod{mn-1}$. Let $k = am + b$ where $0 \leq a \leq n-1$ and $0 \leq b \leq m-1$. We will claim that after a weird shuffle, the card ends up in position $-nk \pmod{mn-1}$.

After the first step, this card will be the $b+1$-th card from top in the $a+1$-th pile. Therefore after the second step, it will be the $bn + a + 1$-th card from bottom - which is position $k' = mn - (bn + a + 1)$, which satisfy $k' = mn - (bn + a + 1) \equiv -bn - anm \equiv -nk \pmod{mn-1}$ as required.

Since $\gcd(n, mn-1) = 1$, after $t = \phi(mn-1)$ rounds, the card in starting position $k \neq 0, mn-1$ will end up in position $(-n)^t k \equiv k \not\equiv 0 \pmod{mn-1}$, which must be the starting position - so after $\phi(mn-1)$ rounds, the cards in position 1 to $mn-2$ will return to their respective position.

Therefore within $\text{lcm}(2, \phi(mn-1))$ shuffles, the deck will return to the initial position. If $mn - 1 = 2$ then $\text{lcm}(2, \phi(mn-1)) = 2 \leq mn - 1$. Else $2 \mid \phi(mn-1)$ so $\text{lcm}(2, \phi(mn-1)) = \phi(mn-1) < mn - 1$. $\square$

Problem 2. Determine all polynomials f with integer coefficients such that there exist an infinite sequence $a_1, a_2, a_3, \ldots$ of positive integers with the property: for all $n \in \mathbb{N}$, the sum of any $f(n)$ consecutive terms of this sequence is divisible by $n+1$. (This requires $f(n) > 0$ for all n.)

Solution. The necessary and sufficient condition is $f(-1) = 0$, in other words, $f(x) = (x+1)g(x)$ for some $g(x) \in \mathbb{Z}[x]$. The infinite sequence of 1 serves as an example of the sequence we want.

Now we assume, for the contrary, that $f(-1) \neq 0$ and there is a sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ satisfying with the condition.

Lemma. If $f(-1) \neq 0$, then there is a positive integer a such that $f(x)$ and $f(ax+a-1)$ have no common factor.

Proof. Assume S is a set of all roots of $f(x)$. Since $f(x) \not\equiv 0$, S is finite. And because $-1 \notin S$, we can find a positive integer a such that $S \cap \{\frac{x+1-a}{a} \mid x \in S\} = \emptyset$. Therefore $f(x)$ and $h(x) = f(ax+a-1)$ have no common root. So their gcd is 1 as desired. □

Back to the problem, by Lemma, we have a positive integer a such that $f(x)$ and $f(ax+a-1)$ have no common factor, so there exists $A(x), B(x) \in \mathbb{Z}[x]$ with positive leading coefficient and positive integer c such that either $A(x)f(x) - B(x)f(ax+a-1) \equiv c$ or $B(x)f(ax+a-1) - A(x)f(x) \equiv c$. Let $T = a_1 + a_2 + \cdots + a_c > 0$. Taking large enough value $k \in \mathbb{Z}^+$ such that $A(k) > 0, B(k) > 0$, and $k > T$, we have $A(k)f(k) - B(k)f(ak+a-1) = c$ or $-c$.

For the case $A(k)f(k) - B(k)f(ak+a-1) = c$, since $k+1$ divides the sum of any $f(k)$ consecutive terms of the sequence, it also divides the sum of any $A(k)f(k)$ consecutive terms of the sequence. Hence $k+1 \mid a_1 + a_2 + \cdots + a_{A(k)f(k)}$. Similarly, we have $k+1 \mid ak+a \mid a_{c+1} + a_{c+2} + \cdots + a_{c+B(k)f(ak+a-1)}$. Since $B(k)f(ak+a-1) + c = A(k)f(k)$, this means that $k+1 \mid a_1 + a_2 + \cdots a_c = T$ which contradicts the fact that $k > T$. The other case can be proved similarly.

Hence we can conclude that the assumption $f(-1) \neq 0$ is false. □

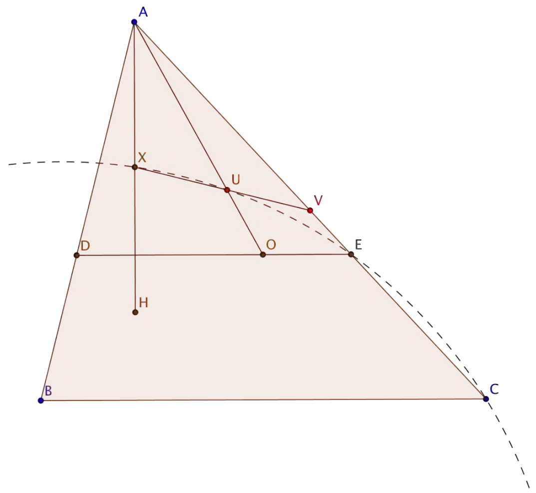
Problem 3. Let $\triangle ABC$ be a triangle with circumcenter O and orthocenter H. The line through O parallel to BC intersect AB at D and AC at E. X is the midpoint of AH. Prove that the circumcircles of $\triangle BDX$ and $\triangle CEX$ intersect again at a point on line AO.

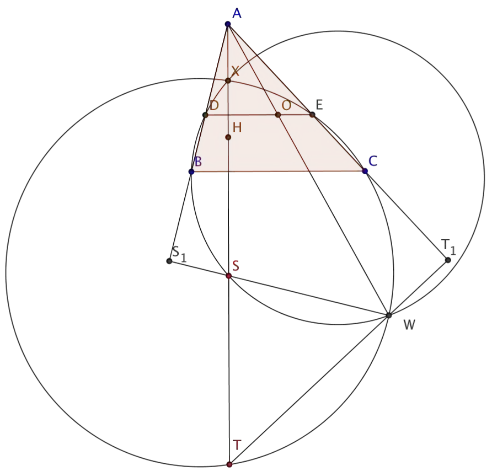
Solution. First we will start with a lemma.

Lemma. If AH, AO cut (CEX) again at T, W both outside $\triangle ADE$ then $TW \perp AC$.

Proof. Let V be the midpoint of AC, VX intersect AO at U. By some side-chasing (not shown here) we have $\frac{VC}{VU} = \frac{\sin C}{\cos B} = \frac{VX}{VE}$, so $U \in (CEX)$.

Note that from $\angle BAH = \angle CAO$ we have $(\triangle AUX, AB) \sim (\triangle ATW, AC)$ where here AB, AC denote straight lines. Since $UX \parallel CH \perp AB$, it follows that $TW \perp AC$.□





Let AH, AO cut (BDX) again at S, W'. Similarly we have $SW' \perp AB$. Now from

$$AT = \frac{AC \cdot AE}{AX} = \frac{AB \cdot \sin^2 B}{AD \cdot AX \cdot \sin^2 C} = AS \cdot \frac{\sin^2 B}{\sin^2 C},$$

we have

$$AW = \frac{AT_1}{\sin B} = \frac{AT \cdot \sin C}{\sin B}$$

and similarly

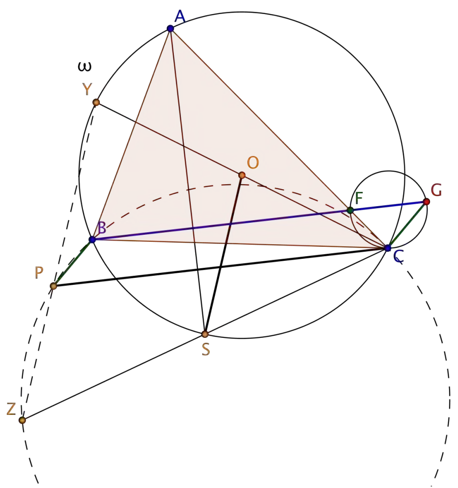
$$AW' = \frac{AS_1}{\sin C} = \frac{AS \cdot \sin B}{\sin C}.$$

Hence $AW = AW'$. Thus $W = W'$, and so $AO, (CEX)$ and (BDX) concur at $W (= Z)$. □

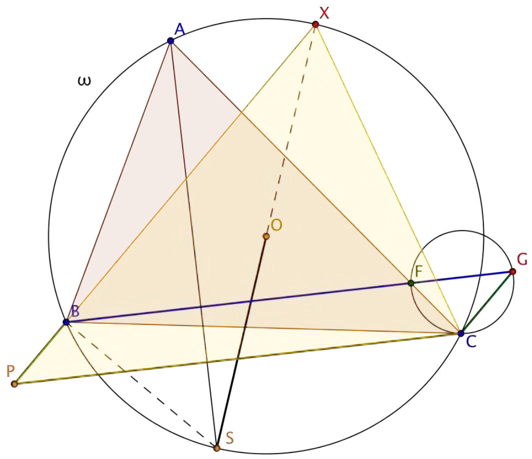
Problem 4. Given an acute triangle $\triangle ABC$ with circumcircle ω and circumcenter O. The symmedian through A intersects ω again at $S \neq A$. Point F is on AC such that $BF \perp AS$, and point G is on ray BF such that $BF \times BG = BC^2$. Finally, let P be the point such that $\square BGCP$ is a parallelogram.

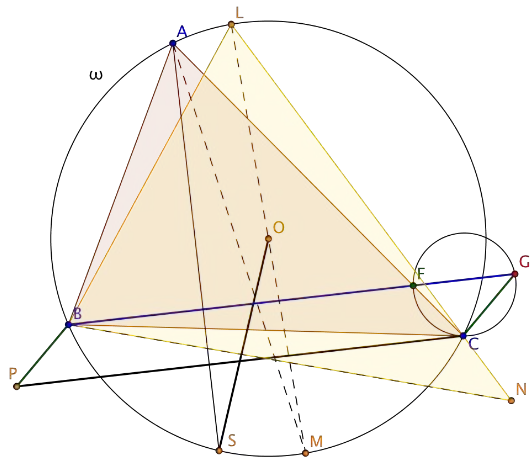
Prove that OS bisects CP.

Solution 1. Let Y, Z be the reflections of C over O, S respectively and M the midpoint of BC. We have $\angle BZC = \angle MSC = \angle ACB$. From $\angle BPC = \angle BGC = \angle ACB$, Z, P, B, C are concyclic, hence $\angle PZC = 180° - \angle PBC = 180° - \angle BCG = 180° - \angle BFC = \angle BFA$. On the other hand, we also have $\angle YZC = \angle OSC = 90° - \angle SAC = \angle BFA$, therefore Y, P, Z are collinear, and by homothety OS must pass through the midpoint of CP. $\square$



Solution 2. Let PB cuts (ABC) at X. We will prove that S is the antipode of X. Since $\angle SBP = \angle CBP - \angle SBC = \angle BCG - \angle SAC = \angle BFC - \angle SAC = \angle(BF, SA) = 90°$. Now, it is easily seen that $\triangle XPC$ is similar to $\triangle ACB$,from $\angle BXS = \angle CAM$ where M is the midpoint of BC from the property of symmedian, we then get that XS bisects CP. $\square$





Solution 3. Reflect S, P over the perpendicular bisector of BC to M, N, and let $MO \cap \omega = \{M, L\}$. Simple angle chasing to get $\triangle BAF \sim \triangle BLC$ so there exist a spiral similarity λ centered at B that sends $\triangle BAF$ to $\triangle BLC$. Since $\frac{FC}{BF} = \frac{CG}{BC} = \frac{CN}{BC}$, we have $\lambda(C) = N$. Since $\angle AML =$ angle of spiral sim; $\lambda(\overleftrightarrow{AM}) = \overleftrightarrow{LM}$, so LM is median of $\triangle BLN$ thus LM bisects BN. Reflect back across perpendicular bisector of BC to get OS bisects CP. $\square$

Problem 5. Suppose that we draw t straight lines through an $n \times n$ table such that for each unit square U in the table, at least one line passes through the *interior* of U.

Prove that $t > (2-\sqrt{2})n$.

Solution. Let the center of the table be the origin so the top-right square has center $\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$. We will ignore squares close to the corner and will only consider squares with centers (x, y) such that $|x|+|y| \leq n-k-1$, where the nonnegative integer k will be chosen later. By doing this we are ignoring at most $4(1+2+\cdots+k) = 2k(k+1)$ squares, so there are at least $n^2 - 2k(k+1)$ squares left. Also, each line can cut through at most $2n-1-2k$ (remaining) unit squares. Therefore the number of lines is at least

$$\left\lceil \frac{n^2 - 2k(k+1)}{2n-1-2k} \right\rceil.$$

Choose $k = \lfloor (1-\frac{1}{\sqrt{2}})n \rfloor$. So $k = (1-\frac{1}{\sqrt{2}})n - \alpha$ for some $\alpha \in [0,1)$, and there are at least

$$\left\lceil \frac{n^2 - 2((1-\frac{1}{\sqrt{2}})n-\alpha)((1-\frac{1}{\sqrt{2}})n+1-\alpha)}{2n-1-2(1-\frac{1}{\sqrt{2}})n+2\alpha} \right\rceil$$

lines, which is equal to

$$\left\lceil \frac{(2\sqrt{2}-2)n^2 + (2\alpha-1)(2-\sqrt{2})n + 2\alpha(1-\alpha)}{\sqrt{2}n + (2\alpha-1)} \right\rceil$$

and this is clearly more than $(2-\sqrt{2})n$, because $\alpha(1-\alpha) > 0$. $\square$

Problem 6. Given a polynomial $P \in \mathbb{R}[x]$ with odd degree. A real number x is called *orbiting* if the sequence

$$x,\ P(x),\ P(P(x)),\ \ldots$$

is bounded. If all orbiting numbers are rational, prove that there are finitely many (or zero) orbiting numbers.

Solution. First we will prove the case $\deg P = 1$.

Let $P(x) = ax + b$. When $a = -1$ or $a = 1$ and $b = 0$ then $P(P(x)) = x$, thus every real number r is orbiting, while if $a = 1$ and $b \neq 0$ then every real number r is not orbiting, so these two cases are trivial. If $|a| \neq 1$, define the function $f(x) = x - \frac{b}{1-a}$. It can be shown that $f(P(x)) = af(x)$, therefore if $|a| > 1$ only $\frac{b}{1-a}$ is orbiting, and if $|a| < 1$ then all reals are orbiting. In all cases, either finitely many numbers are orbiting or all real numbers are orbiting, so we have proven the case $\deg P = 1$.

Henceforth we will assume that $\deg P = n \geq 3$. We will divide our proof for that case into four parts, denoted by P1-P4.

<u>P1: the polynomial P has rational coefficients</u>

Since r being orbiting implies $P(r)$ is too, $P(r)$ must be rational for any orbiting r. We choose $n+1$ orbiting rationals $r_1, r_2, \ldots, r_{n+1}$ and let $P(r_i) = s_i \in \mathbb{Q}$ for each i. Since P is the unique polynomial with $\deg \leq n$ satisfying $P(r_i) = s_i$ for each i, by the Lagrange interpolation formula, $P(x)$ must be the polynomial

$$\sum_{i=1}^{n+1} s_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

which clearly has rational coefficients. □

<u>P2: there exists an interval $[a, b]$ such that all orbiting r lies in it.</u>

Let S be the set containing the (at most $2n$) roots of $|P(x)| = 2|x|$. S must be nonempty, as the polynomial $P(x) - 2x$ have odd degree and thus has a root.

Let $u = \min S$ and $v = \max S$. Choose $[a, b] = [-\max\{|u|, |v|\}, \max\{|u|, |v|\}]$. All numbers not in $[a, b]$ will satisfy $|P(x)| > 2|x|$ or else $|P(x)| \leq 2|x|$ for $x \to \infty$ or $x \to -\infty$ which cannot be true as $|P'(x)| \to \infty$. Therefore if r is not in $[a, b]$, $|P(r)| > 2|r|$ thus $P(r)$ is also not in $[a, b]$. This can be continued indefinitely to obtain $|P^k(r)| > 2^k|r|$ for all positive integers k, thus r cannot be orbiting. Hence all orbiting numbers lies in the interval. □

For P3, we will define some notations: for each rational $r \neq 0$, p_r and q_r are the unique integers satisfying $q_r > 0$, $\gcd(p_r, q_r) = 1$ and $r = \frac{p_r}{q_r}$. For each prime t define the valuation $\nu_t(r)$ as $\nu_t(p_r) - \nu_t(q_r)$. When $r = 0$, define $p_0 = 0$ and $q_0 = 1$ and $\nu_t(0) = +\infty$. We note that the valuation ν has a property: if $\nu_t(r_1) \neq \nu_t(r_2)$ then $\nu_t(r_1 + r_2) = \min\{\nu_t(r_1), \nu_t(r_2)\}$, and if $\nu_t(r_1) = \nu_t(r_2)$ then $\nu_t(r_1 + r_2) \geq \nu_t(r_1)$.

<u>P3: there exists an integer N such that if $q_r > N$ then $q_{P(r)} > q_r$.</u>

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where each a_i is rational. If $\nu_t(a_i) \neq 0$ for some i, then we call the prime t *poisoned*.

Let T be the set of poisoned primes, which is clearly a finite set. For each $t \in T$, let m_t be a positive integer big enough such that

$$-nm_t + \nu_t(a_n) < \min\{-(n-1)m_t + \nu_t(a_{n-1}), -(n-2)m_t + \nu_t(a_{n-2}), \cdots, \nu_t(a_0), -2m_t\}$$

All positive integers greater than m_t will also have the same property. This means that if $\nu_t(r) = -m$ (that is, $\nu_t(q_r) = m$) with $m \geq m_t$ then

$$\nu_t(a_{n-1}r^{n-1} + \cdots + a_1 r + a_0) \geq \min\{-(n-1)m_t + \nu_t(a_{n-1}), -(n-2)m_t + \nu_t(a_{n-2}), \cdots, \nu_t(a_0)\}$$
$$> -nm + \nu_t(a_n) = \nu_t(a_n r^n),$$

implying that $\nu_t(q_{p(r)}) = -\nu_t(P(r)) = -\min\{\nu_t(a_n r^n), \nu_t(a_{n-1}r^{n-1} + \cdots + a_1 r + a_0)\} > 2m$. Note also that for each prime p dividing q_r and not in T, we have $\nu_t(q_{p(r)}) = -\nu_p(P(r)) = n\nu_p(q_r) > 2\nu_p(q_r)$.

Now we show that the integer

$$N = \prod_{t \in T} t^{2m_t}$$

satisfy the required condition. For each integer k, let

$$f(k) = \prod_{t \in T,\ t|k,\ \nu_t(k) < m_t} t^{\nu_t(k)}.$$

It is easily seen that $f(k) < \prod_{t \in T} t^{m_t} < \sqrt{N}$, and that for each prime p dividing $\frac{q_r}{f(q_r)}$, $\nu_p(q_{P(r)}) > 2\nu_p(q_r)$. Thus for each $q_r > N$,

$$q_{P(r)} \geq \prod_{p | \frac{q_r}{f(q_r)}} p^{\nu_p(q_{P(r)})} > \prod_{p | \frac{q_r}{f(q_r)}} p^{2\nu_p(q_r)} > \left(\frac{q_r}{f(q_r)}\right)^2 > \left(\frac{q_r}{\sqrt{N}}\right)^2 > q_r. \qquad \square$$

P4: There are finitely many orbiting numbers r.

Since $n = \deg P$ is odd, P is onto to $\mathbb{R}$. We will define the function $g : \mathbb{R} \to \mathbb{R}$ as follows: for each y, $g(y)$ is the least-valued root of $P(x) = y$. Since r is orbiting, $g^k(r)$ is also orbiting for all positive integer k. If $q_{g^l(r)} > N$ for some $l \geq q_r - N$, then $q_r = q_{P^l(g^l(r))} > N + l \geq q_r$, which is impossible, thus $q_{g^l(r)} \leq N$ for all $l \geq q_r - N$. There are finitely many rational numbers s, say A numbers, that satisfy $s \in [a, b]$ and $q_s \leq N$, and for all $l \geq q_r - N$, $g^l(r)$ is one of those (as it is orbiting and have $q_{g^l(r)} < N$. By the pigeonhole principle, two of $g^m(r), g^{m+1}(r), \ldots, g^{m+A}(r)$ are equal, where $m = q_r - N$. Let these two be $g^{k_1}(r)$ and $g^{k_2}(r)$, with $k_1 > k_2$. The infinite sequence $g^{k_1}(r), P(g^{k_1}(r)), P(P(g^{k_1}(r))), \ldots$ will have the $k_2 - k_1 + 1^{\text{th}}$ term equal to the first term, and therefore must be periodic with period $k_1 - k_2$. As $r = P^k(g^{k_1}(r))$ is in this sequence, $P^{k_1 - k_2}(r) = r$. If $q_r > N$ then $q_{P^{k_1-k_2}(r)} > q_r$ which is a contradiction, thus $q_r \leq N$. Therefore r must also be one of the A rational numbers such that $r \in [a, b]$ and $q_r \leq N$, so there are at most A orbiting numbers. This ends our proof. $\square$